Twisted Topological Hochschild Homology

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October 23, 2025

A thesis submitted for the degree of Bachelor of Mathematical Sciences (Honours)
of the Australian National University

For everyone who shaped who I am today

Declaration

The work in this thesis is my own except where otherwise stated.

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Acknowledgements

My first thanks go to my supervisor, Vigleik Angeltveit. I deeply appreciate the enthusiasm he brings to algebraic topology and mathematics as a whole. I am very grateful for all of the time and effort he put in to help me throughout my honours year. I also thank him for introducing me to the beauty of homotopy theory in his special topics course.

I would like to thank all of my officemates for keeping me accountable and getting me through this year. In particular: Brice for being someone who I can spout abstract nonsense to without him pulling a disgusted face, and Yangda for giving me the profound advice of going from $n = 2$ to setting $n = n$ (which ended up working).

Thanks of course go to my parents, for dealing with the highs of me excitedly telling them about mathematics that sounds like nonsense, and the lows of me realising that it really was nonsense and needing to restart. Thank you Ma, for reading and re-reading every page of my thesis and arguing with me about placements of commas.

Thank you to Hayden and Alison (alphabetical by surname) for being such good friends through this year, and giving your horrified inputs on how mathematics is written. Thank you Jammie for supporting me while I write this thesis, and giving me a Goodliffe.

Finally, thank you to everyone else who has had to suffer through me excitedly talking about things you probably do not care about.

Abstract

We consider a twisted version of topological Hochschild homology for C_n-ring spectra. For computing counit-unit composites, we use an original choice of coset representatives $\{e, \gamma^n, \gamma^{2n}, \ldots, \gamma^{(p-1)n}\}$ for $C_{p^k n}/C_{p^{k-1} n}$ which yields the identity after restriction, mirroring the triangle identity, making this the "categorical" choice. For $H \subset G$, we prove that there is an identification between the twisted cyclic bar construction for G of a norm from H and the subdivided twisted cyclic bar construction for H: $i_H^G B_\bullet^{cy,G}(N_H^G R) \cong sd_m B_\bullet^{cy,H}(R)$. Finally, we propose a definition of C_n-twisted TC_{C_n} and prove that this definition can be identified with a homotopy equaliser.

Contents

Notation and terminology

In the following, X, Y can be interpreted as topological spaces or spectra.

Notation

$*$	The space consisting of a single point.
I	The closed interval $[0,1]$.
$X \sqcup Y$	The disjoint union of X and Y.
X_+	The disjoint union of X and a point $*$.
S^n	The n-dimensional sphere, $S^n := \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$.
$X \vee Y$	The wedge sum of X and Y.
$X \wedge Y$	The smash product of X and Y.
$A \cong B$	A and B are isomorphic in the appropriate category.
$X \simeq Y$	Spaces or spectra X and Y are homotopy equivalent.
$X \simeq_G Y$	Spaces or spectra X and Y are G-homotopy equivalent.
e	The trivial group.
C_n	The multiplicative group generated by $e^{2\pi i/n}$.

Terminology

Space	Compactly generated weak Hausdorff topological space.

Chapter 1

Introduction

A first insight in to why one may study homotopy theory comes from Whitehead's theorem [46] stating that spaces can be identified up to homotopy by their homotopy groups. In many situations however, the objects with homotopy properties we wish to study (such as spaces, spectra or simplicial objects) carry natural group actions. In classical "nonequivariant" homotopy theory, these actions are typically ignored, such that we study only the underlying objects. However, by discarding the equivariant structure, we may lose valuable information. This is fixed using equivariant homotopy theory, in so far as it takes group actions into account, thereby providing a richer and more accurate framework for understanding homotopy types. Equivariant methods have played a central role in resolving deep problems in topology, most notably, Hill, Hopkins and Ravenel's solution of the Kervaire invariant one problem in [22], where equivariant techniques were required, despite the problem being originally formulated without explicit reference to equivariant homotopy. This emphasises that equivariant tools can be the hidden structure necessary to unlock otherwise intractable problems.

Classically, topological Hochschild homology (THH) refines Hochschild homology from rings to ring spectra and, via its cyclotomic structure, feeds into topological cyclic homology (TC) and ultimately into computations in algebraic K-theory. Motivating this theory, THH and TC serve as trace approximations to algebraic K-theory.

The purpose of this thesis is to assemble the homotopy theory and equivariant foundations so as to move from THH to a combined version called *twisted* THH. Throughout, we detail structural results and concrete computations to illustrate the objects involved. The exposition is organised to move from fundamentals to applications: homotopy and stable techniques, their equivariant refinements, THH and finally twisted THH and twisted TC. We aim to follow the framework of Bohmann, Gerhardt, Krulewski, Petersen, and Yang (BGKPY) in [7] by introducing the prerequisite knowledge for their paper in Chapters 2, 3, and 4 and commenting on subtleties in Chapter 5.

Following the treatment of BGKPY, we introduce the twisted cyclic bar construction for C_n-ring spectra and give a construction of twisted THH and its algebraic analogue. The authors make an explicit choice of coset representatives in their Lemma 6.10. Their choice of coset representatives aligns with the twisted cyclic bar construction allowing for simplification of their arguments; however, there are other natural choices. From category theory, we expect the specific formula in Lemma 6.10 for the composite of units and counits to be the identity. Using their choice does not yield the identity. In this thesis, we make two technical contributions in order to simplify and contribute to the discussions and proofs in their paper. We show that a different choice of coset representatives $\{e, \gamma^n, \gamma^{2n}, \ldots, \gamma^{(p-1)n}\}$ for $C_{p^k n}/C_{p^{k-1}n}$ makes the unit-counit composite act as the identity after restriction, mirroring the triangle identity, making this the "categorical" choice. We then go on to prove Theorem 5.1.11 with the authors' choice of coset representatives which gives an identification $i_H^G B_\bullet^{cy,G}(N_H^G R) \cong sd_m B_\bullet^{cy,H}(R)$. Finally, we propose an original definition of C_n-twisted TC_{C_n} in Definition 5.2.3 and give Theorem 5.2.5 reconciling this with the nontwisted case.

Throughout the thesis, we often make reference to spaces without any mention of exactly the types of spaces we are considering. Almost nothing is true in the category of all topological spaces. Instead, we make the mild restriction to compactly generated weak Hausdorff spaces. Therefore, when making a claim for all space, in actuality we are referencing all spaces with this mild restriction.

An outline of the chapters of this thesis is as follows:

Chapter 2 – Homotopy Basics. We review core tools in homotopy theory and its stable counterpart, including homotopy groups, Whitehead's theorem, covering space methods, fibrations, and the Serre spectral sequence, before turning to stable constructions and categorical background that will be used throughout.

Chapter 3 – Equivariant Background. We develop the genuine G-equivariant setting: equivariant homotopy and stable homotopy theory, Mackey, Green, and Tambara functors, and discussion of fixed-points. Core discussions include the tom Dieck splitting, which decomposes categorical fixed points in terms of Weyl-quotients and fixed-point spaces, and a systematic treatment of homotopy orbits/fixed points, geometric fixed points, and the Tate construction that underpins cyclotomic structures. These constructions prepare the groundwork for discussions of THH and TC in later chapters.

Chapter 4 – Topological Hochschild Homology. We move on from the algebraic Hochschild homology to THH by replacing tensor with smash and rings with ring spectra via the cyclic bar construction. We then give a worked computation of $\mathrm{THH}(\mathbb{F}_p)$, discuss a Thom spectrum perspective, and introduce TC together with its relationship to Witt vectors on π_0.

Chapter 5 - Twisted Topological Hochschild Homology. Following discussion of BGKPY, we introduce the twisted cyclic bar construction for C_n-ring spectra from [7] and define twisted $\mathrm{THH}_{C_n}(R)$ as its geometric realisation (or equivalently via a norm-model). We then develop an algebraic analogue using twisted cyclic nerves of Mackey functors. We adopt and discuss a "categorical" choice of coset representatives $\{e, \gamma^n, \gamma^{2n}, \ldots, \gamma^{(p-1)n}\}$ for $C_{p^k n}/C_{p^{k-1}n}$, which makes norm unit-counit composites behave like formal triangle identities after restriction. Further to this, we prove Theorem 5.1.11, identifying $i_H^G B_\bullet^{cy,G}(N_H^G R) \cong sd_m B_\bullet^{cy,H}(R)$. We go to define C_n-twisted p-cyclotomic spectra and discuss the subtleties of THH_{C_n} being p-cyclotomic. Finally we propose an original definition of C_n twisted TC_{C_n} and Theorem 5.2.5.

As prerequisites, we assume point-set topology at the level needed for compactly generated weak Hausdorff spaces. We assume familiarity with basic algebraic topology at the level of Hatcher's books [19] and [20] and basic category theory at the level of MacLane's book [26, Ch III, IV, V, VII, X, XI], including (symmetric) monoidal closed categories. The equivariant material is fairly self-contained in Chapter 3, and we review simplicial preliminaries for THH in Chapter 4. Readers new to Witt vectors will find a short primer at the end of Chapter 4.

We propose an original definition of C_n-twisted TC_{C_n} in Definition 5.2.3 and state and prove Theorem 5.2.5. We discuss the subtleties of a p-cyclotomic structure on twisted THH and in defining twisted TC compatible with the ϕ_p twist. The proof of Theorem 4.1.9 completes the sketch in [9]. To the best of the author's knowledge, the formulation and proof of Theorem 5.1.11 in this generality, and the coset choice simplification outlined in Chapter 5, are original. Theorems in Chapter 2 and Theorems 3.1.9, 3.6.2, 4.3.8, 4.5.6, 4.5.7 and 5.1.10 are proved here, though they are well documented. Example 3.5.10 is computed in full detail and the examples of $\mathrm{THH}(\mathbb{F}_p)$ and $\mathrm{TC}(F_p)$ are computed almost in full. All other results are standard in the literature and are cited appropriately.

Chapter 2

Homotopy Basics

This chapter collects the basic homotopy theoretic tools that will be needed later. We begin with homotopy and stable homotopy groups and tools for computing them.

2.1 Homotopy Theory

Homotopy groups detect "higher dimensional holes" in a way that is easier to define than (co)homology groups, but usually more difficult to compute. The motivation for studying homotopy groups comes from Whitehead's Theorem 2.1.2.

Definition 2.1.1. Let (X, x_0) be a based topological space with basepoint x_0. For $k \geq 1$, the *k-th homotopy group* $\pi_k(X, x_0)$ of X based at x_0 is the group of homotopy classes of basepoint preserving maps from $S^k \to X$. We can also think of these as maps from $I^k \to X$ that map the faces to the basepoint $\partial I^k \to x_0$. For $k = 0$, we write $\pi_0(X)$ for the set of path components of X.

Sometimes we shorten this to just $\pi_k(X)$, if either the basepoint is clear from context, or if the choice of basepoint is irrelevant to the argument.

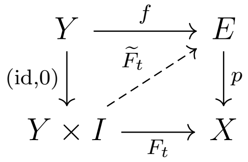
We know that if two spaces X, Y are homotopy equivalent, then they have isomorphic cohomology groups and rings. However, the converse implication does not hold. In order to have a sufficient condition for homotopy equivalences, we need to extend to homotopy theory. This is shown through Whitehead's Theorem:

Theorem 2.1.2 (Whitehead, [46]/[19, Thm. 4.5])**.** *A based map $f\colon X \to Y$ of based CW complexes is a homotopy equivalence if and only if for all $k \in \mathbb{N}$ and $x_0 \in X$, f induces isomorphisms*

$$f_*\colon \pi_k(X, x_0) \xrightarrow{\cong} \pi_k(Y, f(x_0)).$$

We now give background on some methods to compute homotopy groups.

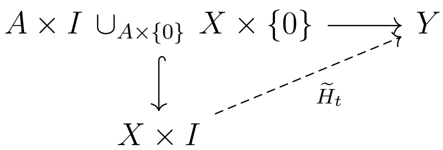
Definition 2.1.3. Let E, X and Y be based spaces. A map $p\colon E \to X$ satisfies the *homotopy lifting property* if given a map $f\colon Y \to E$ and a homotopy F_t of $\overline{f} = p \circ f$ in X, there is a homotopy $\widetilde{F}_t$ of f in E which covers F_t. This is to say that $p \circ \widetilde{F}_t = F_t$. This property may be expressed in terms of the diagram below.



A map is called a *fibration* if it satisfies the homotopy lifting property.

Dually, we have cofibrations.

Definition 2.1.4. A map $i\colon A \hookrightarrow X$ of based spaces satisfies the *homotopy extension property* if for every based space Y, every based map $f\colon X \to Y$ and every homotopy $H_t\colon A \times I \to Y$ with $H_0 = f|_A$, there exists an extension $\widetilde{H}_t\colon X \times I \to Y$ such that $\widetilde{H}_t|_{A\times I} = H_t$ and $\widetilde{H}_0 = f$. This property may be expressed in terms of the diagram below



A map is called a *cofibration* if it satisfies the homotopy extension property.

Theorem 2.1.5. *A covering space $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ induces isomorphisms*

$$p_*\colon \pi_n(\widetilde{X}, \widetilde{x}_0) \xrightarrow{\cong} \pi_n(X, x_0) \quad \textit{for all } n \geq 2.$$

Proof sketch. For surjectivity, apply the lifting criterion which implies every map $(S^n, s_0) \to (X, x_0)$ lifts to $(\widetilde{X}, \widetilde{x}_0)$ for simply connected S^n (so $n \geq 2$). For injectivity, use the homotopy lifting property. □

Theorem 2.1.2 requires the isomorphisms being induced by a map. We give an example exhibiting this requirement.

Example 2.1.6. Consider $X = S^2 \times \mathbb{R}P^3$ and $Y = \mathbb{R}P^2 \times S^3$. Then, $\pi_1(X) = \mathbb{Z}/2 = \pi_1(Y)$, and both X, and Y have the same universal cover $S^2 \times S^3$, so by Theorem 2.1.5, $\pi_k(X) \cong \pi_k(Y)$ for all $k \in \mathbb{N}$. However, we can show that X and Y are not homotopy equivalent. Recall that with $\mathbb{Z}$ coefficients, $H_2(\mathbb{R}P^2) = 0$

and $H_2(\mathbb{R}P^3) = 0$. By the Künneth theorem for homology we compute $H_2(X)$ and $H_2(Y)$.

$$\begin{aligned} H_2(X) &= H_2(S^2 \times \mathbb{R}P^3) \\ &\cong (H_1(S^2) \otimes H_1(\mathbb{R}P^3)) \oplus (H_0(S^2) \otimes H_2(\mathbb{R}P^3)) \oplus (H_2(S^2) \otimes H_0(\mathbb{R}P^3)) \\ &\cong (0 \otimes \mathbb{Z}/2) \oplus (\mathbb{Z} \otimes 0) \oplus (\mathbb{Z} \otimes \mathbb{Z}) \\ &\cong \mathbb{Z}. \end{aligned}$$

On the other hand,

$$\begin{aligned} H_2(Y) &= H_2(\mathbb{R}P^2 \times S^3) \\ &\cong (H_1(\mathbb{R}P^2) \otimes H_1(S^3)) \oplus (H_0(\mathbb{R}P^2) \otimes H_2(S^3)) \oplus (H_2(\mathbb{R}P^2) \otimes H_0(S^3)) \\ &\cong (\mathbb{Z}/2 \otimes 0) \oplus (\mathbb{Z} \otimes 0) \oplus (0 \otimes \mathbb{Z}) \\ &\cong 0. \end{aligned}$$

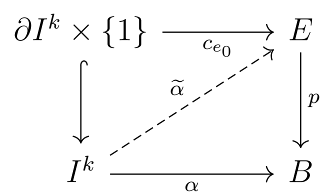
Since $H_2(X) \ncong H_2(Y)$, X and Y are not homotopy equivalent.

Similar to a short exact sequence behaving well with (co)homology, fibrations behave well with homotopy.

Theorem 2.1.7. *Let $p\colon (E, e_0) \to (B, b_0)$ be a basepoint-preserving fibration, and set $F := p^{-1}(b_0)$ with basepoint $e_0 \in F$. Write $i\colon F \hookrightarrow E$ for the inclusion. Then there is a natural long exact sequence*

$$\cdots \longrightarrow \pi_k(F, e_0) \xrightarrow{i_*} \pi_k(E, e_0) \xrightarrow{p_*} \pi_k(B, b_0) \xrightarrow{\partial} \pi_{k-1}(F, e_0) \longrightarrow \cdots$$

Proof sketch. We construct the connecting map ∂ and then indicate exactness. Represent a class $[\alpha] \in \pi_k(B, b_0)$ by a map $\alpha\colon (I^k, \partial I^k) \to (B, b_0)$, so $\alpha|_{\partial I^k} \equiv b_0$. Let $c_{e_0}\colon \partial I^k \to E$ be the constant map at e_0, then $p \circ c_{e_0} = \alpha|_{\partial I^k}$. By the homotopy lifting property for the pair $(I^k, \partial I^k)$, there exists a lift $\widetilde{\alpha}\colon I^k \to E$ with $p \circ \widetilde{\alpha} = \alpha$ and $\widetilde{\alpha}|_{\partial I^k} = c_{e_0}$, shown in the diagram below.



Since the top-face $\partial I^k \times \{1\} \subset I^k$, we have $p \circ \widetilde{\alpha}(x, 1) = \alpha(x, 1) = b_0$. Hence $\widetilde{\alpha}(\cdot, 1)$ lands in F. Define

$$\partial[\alpha] := \left[\widetilde{\alpha}(\cdot, 1)\colon (I^{k-1}, \partial I^{k-1}) \to (F, e_0)\right] \in \pi_{k-1}(F, e_0).$$

We now show that this is well-defined. If $\alpha \simeq \alpha'$ relative to ∂I^k, apply the homotopy lifting property to obtain a homotopy $\widetilde{\alpha} \simeq \widetilde{\alpha}'$ relative to ∂I^k. Restricting to $\partial I^k \times \{1\}$ shows $\partial[\alpha] = \partial[\alpha']$. Next, if $\widetilde{\alpha}$ and $\widetilde{\alpha}'$ are two lifts with the same boundary c_{e_0}, by uniqueness of lifts up to fiberwise homotopy, they are homotopic relative to ∂I^k, so their top-face restrictions are homotopic in F. Thus ∂ is well-defined.

Now we give some indications of exactness. At $\pi_k(E)$: If $i_*[\beta] \in \pi_k(E)$ with $\beta\colon I^k \to F$, then $p_*(i_*[\beta]) = [p \circ \beta] = 0$. Conversely, if $p_*[\gamma] = 0$, then $p \circ \gamma \simeq b_0$ relative to the boundary. Lift this homotopy to deform γ into F, showing $[\gamma] \in \operatorname{im}(i_*)$. Then at $\pi_k(B)$: If $[\alpha] \in \ker \partial$, choose a lift $\widetilde{\alpha}$ as above whose top-face restriction is null-homotopic in F. Glue this null-homotopy to obtain a map $I^k \to E$ lifting α, hence $[\alpha] \in \operatorname{im}(p_*)$. Conversely, if $[\alpha] = p_*[\gamma]$, take $\widetilde{\alpha} = \gamma$, then $\partial[\alpha] = 0$.

The remaining verifications are standard and give exactness at every stage. □

We now introduce some spaces used in common fibrations for computing homotopy groups.

Definition 2.1.8. The *path space* of X is the space PX consisting of all the paths in X with initial point at the basepoint, $PX \coloneqq \mathrm{Map}_*(I, X)$. The *loop space* of X is the space ΩX consisting of all the loops in X with initial point at the basepoint, $\Omega X \coloneqq \mathrm{Map}_*(S^1, X)$. The *free loop space* of X is the space LX consisting of all loops in X, $LX = \mathrm{Map}(S^1, X)$. The *(reduced) suspension* of X is given by an identification of the double cone over X:

$$\Sigma X = (X \times I) \Big/ \big((X \times \{0\}) \cup (X \times \{1\}) \cup (\{x_0\} \times I)\big).$$

We can equivalently write the suspension as $\Sigma X = X \wedge S^1$, and with this suspension definition, can further define $\Sigma^i = \Sigma(\Sigma^{i-1} X) = X \wedge S^i$ since $S^{i-1} \wedge S^1 \cong S^i$. Also, PX is contractible, by retracting every path back to the basepoint.

Theorem 2.1.9 ([36, Prop. A.2.3])**.** *There is an adjunction*

$$\Sigma^i\colon \mathrm{Top}_* \rightleftarrows \mathrm{Top}_*\colon \Omega^i$$

for each $i \in \mathbb{N}$. In particular, $\pi_{k+i}(Y) \cong \pi_k(\Omega^i Y)$.

Combining this theorem with Definition 2.1.8, we can equivalently write $\Omega^i X$ as the space of based maps from $S^i \to X$, $\Omega^i X = \mathrm{Map}_*(S^i, X)$.

With these definitions, there is a natural projection $p\colon PX \to X$ given by the endpoint of a path. From this, we get the pathspace fibration: $\Omega X \to PX \to X$.

Theorem 2.1.10. *$\Omega X \to PX \to X$ is a fibration.*

Proof. We prove the homotopy lifting property. Let Y be any space, let $H\colon Y \times I \to X$ be a homotopy, and suppose we are given a lift $f\colon Y \to PX$ with $p \circ f = H(\,\cdot\,, 0)$. Equivalently, for each $y \in Y$ we have a path $f(y)\colon I \to X$ with $f(y)(0) = x_0$ and $f(y)(1) = H(y, 0)$. Define a map $\widetilde{H}\colon Y \times I \longrightarrow PX$ by specifying, for $(y, s) \in Y \times I$, the path $\widetilde{H}(y, s)\colon I \to X$ as the concatenation

$$\widetilde{H}(y,s)(t) = \begin{cases} f(y)(2t) & 0 \le t \le \frac{1}{2}, \\ H\big(y, (2t-1)s\big) & \frac{1}{2} \le t \le 1. \end{cases}$$

This is continuous (with the compact-open topology) and starts at x_0 for all (y, s) since $\widetilde{H}(y, s)(0) = f(y)(0) = x_0$. At $t = \frac{1}{2}$ the two pieces agree because $f(y)(1) = H(y, 0)$, by the assumption $p \circ f = H(\,\cdot\,, 0)$.

By construction, $p\big(\widetilde{H}(y, s)\big) = \widetilde{H}(y, s)(1) = H(y, s)$ and $\widetilde{H}(y, 0) = f(y)$. Hence $\widetilde{H}$ is a lift of H starting at f, so p satisfies the homotopy lifting property. Therefore p is a fibration. Finally, the fiber over x_0 consists of paths α with $\alpha(0) = \alpha(1) = x_0$. That is, loops based at x_0. Thus $p^{-1}(x_0) = \Omega X$. □

It turns out that (co)homology can be computed from a fibration as well. The *Serre spectral sequence* computes (co)homology of the total space of a fibration from the (co)homology of the base and fiber. It also encodes how $\pi_1(B)$ acts on the (co)homology of the fiber.

Theorem 2.1.11 ([20, §5.1])**.** *Let $F \to E \xrightarrow{p} B$ be a fibration with B simply connected and CW. There are first quadrant spectral sequences with signatures*

$$E^2_{s,t} \cong H_s(B; H_t(F)) \implies H_{s+t}(E),$$
$$E_2^{s,t} \cong H^s(B; H^t(F)) \implies H^{s+t}(E)$$

natural in maps of fibrations.

For B not simply connected, coefficients are twisted by the $\pi_1(B)$-action. For cohomology with a commutative ring of coefficients, the Serre spectral sequence is multiplicative. That is, d_r is a derivation and $E_2^{*,*}$ identifies with the cohomology of B with coefficients in the graded-commutative algebra $H^*(F)$. This is useful for tracking product structures.

Example 2.1.12. Consider the pathspace fibration $\Omega X \to PX \to X$. Since PX is contractible, $H^n(PX) = 0$ for $n > 0$, so all classes in $E_2^{s,t} \cong H^s(X; H^t(\Omega X))$ must be killed by differentials. This will be useful in computations in Chapter 4.

2.2 Stable Homotopy Theory

Homotopy groups can be very difficult to compute. We instead change perspective and compute the *stable* homotopy groups. This stability is made precise in the following:

Theorem 2.2.1 (Freudenthal Suspension, [19, Cor. 4J.3])**.** *Let X be an n-connected based space. Then the map $X \to \Omega(\Sigma X)$ induces a map $\pi_k(X) \to \pi_k(\Omega(\Sigma X)) \cong \pi_{k+1}(\Sigma X)$ which is an isomorphism if $k \leq 2n$.*

Example 2.2.2. Since S^n is $(n-1)$-connected, by Theorem 2.2.1, $\pi_{n+k}(S^n) \cong \pi_{m+k}(S^m)$ for $n, m \geq k+2$.

Theorem 2.2.1 implies that in the sequence of iterated suspensions

$$\pi_k(X) \to \pi_{k+1}(\Sigma X) \to \pi_{k+2}(\Sigma^2 X) \to \cdots$$

all maps are eventually isomorphisms. For any X note that Σ increases the connectivity of X by one, so the maps are eventually isomorphisms, even without any connectivity assumptions on X. With this, we can then define the stable homotopy groups based on this eventual isomorphism.

Definition 2.2.3. The *stable homotopy groups* of X are

$$\pi_k^s(X) \coloneqq \varinjlim_j \pi_{k+j}\left(\Sigma^j X\right), \quad \text{for } k \in \mathbb{Z}.$$

One may think that a simple space to consider would be $X = S^0$. However, this turns out to be a deep and complicated problem to compute the stable homotopy groups of S^0.

Example 2.2.4. Consider $X = S^0$. Since $\Sigma^j S^i \cong S^{i+j}$, the definition $\pi_k^s(X) = \varinjlim_j [S^{k+j}, \Sigma^j X]$ gives $\pi_m^s(S^n) \cong \pi_{m-n}^s(S^0)$. We know that $\pi_k(S^k) \cong \mathbb{Z}$, so we get $\pi_0^s(S^0) \cong \mathbb{Z}$. For $k \geq 1$ the groups $\pi_k^s(S^0)$ are generally very complicated.

Note that *a priori*, we could have negatively graded stable homotopy groups, since we are only looking at the eventual isomorphisms.

Corollary 2.2.5. *Let X be a space, then $\pi_k^s(X)$ is concentrated in $k \geq 0$.*

Proof. Let $k < 0$, we have $\pi_{k-k}(\Sigma^{-k} X) = \pi_0(\Sigma^{-k} X) = 0$. Since $\Sigma^{-k} X$ is at least $(-k-1)$-connected, by Theorem 2.2.1, $\pi_0(\Sigma^{-k} X) \to \pi_1(\Sigma^{-k+1} X)$ is an isomorphism. Reapplying this theorem at each step gives that $\pi_{k+j}(\Sigma^j X) = 0$ for all $j \geq -k$. □

When we study stable homotopy theory, it turns out to be richer to work with objects called *spectra* instead of spaces. We will see that the homotopy groups of some spectra can be associated to the stable homotopy groups of a space.

Definition 2.2.6. A *sequential spectrum* X consists of the following data:

- based spaces X_n for each $n \in \mathbb{N}$,
- based maps $\sigma_n \colon \Sigma X_n \to X_{n+1}$.

A map of sequential spectra $f \colon X \to Y$ is a sequence of maps $f_n \colon X_n \to Y_n$ such that the diagram below commutes:

$$\begin{array}{ccc} \Sigma X_n & \xrightarrow{\Sigma f_n} & \Sigma Y_n \\ \downarrow{\scriptstyle \sigma_n} & & \downarrow{\scriptstyle \sigma_n} \\ X_{n+1} & \xrightarrow{f_{n+1}} & Y_{n+1} \end{array}$$

We give some examples of basic sequential spectra.

Example 2.2.7. The *sphere spectrum*, denoted $\mathbb{S}$, is a sequential spectrum where the n-th space is S^n. Each S^n is a based space, and since $\Sigma S^n \cong S^{n+1}$, there is the canonical map $\sigma_n \colon \Sigma S^n \xrightarrow{\cong} S^{n+1}$.

Example 2.2.8. Let X be a based space. The *suspension spectrum* of X, denoted $\Sigma^\infty X$, is a sequential spectrum where the n-th space is $(\Sigma^\infty X)_n \cong \Sigma^n X$. Each $\Sigma^n X$ is a based space, and trivially we have maps $\sigma_n \colon \Sigma\Sigma^n X \xrightarrow{\cong} \Sigma^{n+1} X$.

It turns out that we cannot define a symmetric monoidal product on sequential spectra, which is desirable for when we go equivariant. To fix this, we look at more structured spectra.

Definition 2.2.9. An *orthogonal spectrum* X consists of the following data:

- based spaces X_n with $O(n)$-action for each $n \in \mathbb{N}$,
- based maps $\sigma_{m,n} \colon \Sigma^m X_n \to X_{n+m}$ that are $O(n) \times O(m) \subset O(n+m)$-equivariant.

A map of orthogonal spectra is a collection of $O(n)$–equivariant maps $f_n \colon X_n \to Y_n$ compatible with the structure maps.

To define a product, it is convenient to take a step back to orthogonal sequences and their external smash product.

Definition 2.2.10. An *orthogonal sequence* X consists only of the data of based spaces $\{X_n\}_{n\in\mathbb{N}}$ with an $O(n)$-action on X_n. Define the *external smash product* of two orthogonal sequences to be

$$(X\bar{\wedge}Y)_n = \bigvee_{i+j=n} O(n)_+ \wedge_{O(i)\times O(j)} X_i \wedge Y_j.$$

Considering $\mathbb{S}$ as an orthogonal sequence, an *orthogonal spectrum* X is an $\mathbb{S}$-module. For orthogonal spectra X, Y, define the *smash product* $X \wedge Y$ as the coequaliser: $X\bar{\wedge}\mathbb{S}\bar{\wedge}Y \rightrightarrows X\bar{\wedge}Y \longrightarrow X \wedge Y.$

To see that the definitions of an orthogonal spectrum agree, let X be an orthogonal sequence. Since $(X\bar{\wedge}\mathbb{S})_n = \bigvee_{i+j=n} O(n)_+ \wedge_{O(i)\times O(j)} X_i \wedge S^j$, a map $\sigma_n\colon (X\bar{\wedge}\mathbb{S})_n \to X_n$ is the same information as $O(n)$-maps $O(n)_+ \wedge_{O(i)\times O(j)} X_i \wedge S^j \to X_n$. Then by definition, these maps are the same information as $O(i)\times O(j)$-maps $X_i \wedge S^j \to X_n$, the required maps from Definition 2.2.9 of orthogonal spectra.

The external smash product is visibly symmetric monoidal, as is the smash product defined above, due to [29]. We could consider a similar construction for sequential spectra, but this turns out to not be symmetric monoidal. For X, Y sequences of based spaces $\{X_n\}_{n\in\mathbb{N}}$ and $\{Y_n\}_{n\in\mathbb{N}}$, and external smash $(X\bar{\wedge}Y)_n = \bigvee_{i+j=n} X_i \wedge Y_j$, define the smash product in the same way and consider $\mathbb{S}\wedge\mathbb{S}$. The symmetry map $\tau\colon \mathbb{S}\wedge\mathbb{S} \to \mathbb{S}\wedge\mathbb{S}$ at the second level will be $S^1 \wedge S^1 \xrightarrow{\text{swap}} S^1 \wedge S^1 \cong S^2$. We know that this is a degree -1 map, but that means that τ cannot be the identity, so the product is not symmetric.

Remark 2.2.11. For the remainder of this thesis, when we take the smash product $X \wedge Y$ of orthogonal spectra, we mean a derived version of the smash product. Equivalently, taking the smash product after doing a cofibrant (to be defined in Definition 2.3.2) replacement on the spectra. This is analogous to using the derived tensor product or taking a projective resolution in algebra.

Most of the spectra we care about and use turn out to be orthogonal spectra.

Example 2.2.12. Following on from Example 2.2.7, the sphere spectrum $\mathbb{S}$ is an orthogonal spectrum. To see this, consider S^n as the 1-point compactification of $\mathbb{R}^n$, then $O(n)$ acts on $\mathbb{R}^n \cup \{\infty\} \simeq S^n$ with $\{\infty\}$ the basepoint.

In what follows, when we say spectrum, we mean orthogonal spectrum, and we denote the category of orthogonal spectra simply by Sp.

Definition 2.2.13. A *ring spectrum* A is a spectrum equipped with a multiplication map $\mu\colon A \wedge A \to A$ and a unit map $\eta\colon \mathbb{S} \to A$ which satisfies (strict) associativity and unital conditions: $\mu(\mathrm{id} \wedge \mu) = \mu(\mu \wedge \mathrm{id})$, and $\mu(\mathrm{id} \wedge \eta) = \mathrm{id} = \mu(\eta \wedge \mathrm{id})$. A *module spectrum* is a module over a ring spectrum.

We can consider the swap map τ which swaps the smash factors in $A \wedge A$. We say that a ring spectrum is (strictly) commutative if $\mu\tau = \mu$. One could define ring spectra which satisfy the associativity, unital and commutativity conditions only up to homotopy, though we only deal with the strict versions.

Example 2.2.14. Following on from Example 2.2.12, the sphere spectrum $\mathbb{S}$ is a ring spectrum with $S^n \wedge S^m \cong S^{n+m}$. In particular, $\mathbb{S} \wedge \mathbb{S} \cong \mathbb{S}$.

We now want to do homotopy theory with spectra.

Definition 2.2.15. Let X be a spectrum. The *homotopy groups* of X are given by a colimit over the groups,

$$\pi_n(X) \coloneqq \varinjlim \pi_{n+k}(X_k).$$

It is common to just write $\pi_n X$, though we make an effort to write $\pi_n(X)$ when this helps clarify order of operations.

We can see that this extends the stable homotopy groups of spaces:

Example 2.2.16. Let X be a based space, by definition of the suspension spectrum, $\pi_n^s(X) = \pi_n(\Sigma^\infty X)$. For $X = S^0$, we know by definition that $\Sigma^\infty S^0 = \mathbb{S}$, so we see $\pi_n^s(S^0) = \pi_n(\mathbb{S})$. For this reason, it is common to see S^0 in place of $\mathbb{S}$ in the literature.

Theorem 2.2.17 ([27])**.** *The category* Sp *of (orthogonal) spectra is a closed symmetric monoidal category with unit* $\mathbb{S}$*. For every spectrum* Y*, the functor* $(-) \wedge Y\colon \mathrm{Sp} \to \mathrm{Sp}$ *has a right adjoint* $F(Y, -)\colon \mathrm{Sp} \to \mathrm{Sp}$*, called the* internal function spectrum*. Moreover,* Sp *is enriched, tensored, and cotensored over based spaces: for a based space* A *and a spectrum* X*, the tensor is* $X \wedge A$ *and the cotensor is* $\mathrm{Map}_*(A, X)$*.*

Here $(X \wedge A)_n = X_n \wedge A$ and $\mathrm{Map}_*(A, X)_n = \mathrm{Map}_*(A, X_n)$, with structure maps induced from X.

Proof sketch of Theorem 2.2.17. In orthogonal spectra, the smash product $\wedge$ is constructed so that $(\mathrm{Sp}, \wedge, \mathbb{S})$ is symmetric monoidal and the function object

$F(-,-)$ is defined so that the $(\wedge, F)$ adjunction holds. The enrichment over based spaces comes from the mapping space $F(X,Y)$, built levelwise with compatibility with structure maps, and the tensor/cotensor with a based space A are given by $X \wedge A$ and $\mathrm{Map}_*(A,X)$ (levelwise), yielding the adjunctions. □

Therefore, for a spectrum X, we can define $\Sigma^n X := X \wedge S^n$ and $\Omega^n X := \mathrm{Map}_*(S^n, X)$ as we did with spaces.

Lemma 2.2.18 ([29, Thm. 0.1])**.** *There are stable equivalences $X \xrightarrow{\simeq} \Omega^n\Sigma^n X$ and $\Sigma^n\Omega^n X \xrightarrow{\simeq} X$.*

With this lemma, for $n < 0$ we abuse notation to define $\Sigma^n X := \Omega^{-n} X$.

We now introduce some fundamental spectra that are used throughout stable homotopy theory.

Example 2.2.19. Let $X = HG$ be an *Eilenberg-MacLane spectrum* so that

$$\pi_n(HG) \cong \begin{cases} G & n = 0, \\ 0 & \text{else.} \end{cases}$$

We can build a model for such a spectrum for any abelian group G. We have X_n a CW complex $K(G,n)$ and $\sigma\colon \Sigma K(G,n) \to K(G,n+1)$ given by the adjoint of the map $K(G,n) \xrightarrow{\simeq} \Omega K(G,n+1)$.

Remark 2.2.20. If R is a ring, then HR naturally admits the structure of a ring spectrum. This provides a bridge from algebra to spectra and lets us extend algebraic invariants to the spectral setting (see Chapter 4).

Something interesting about spectra is that unlike spaces as in Lemma 2.2.5, the homotopy groups of spectra need not be concentrated in nonnegative degrees.

Example 2.2.21. Let $X = \Sigma^i HG$, then

$$\pi_n(\Sigma^i HG) \cong \begin{cases} G & n = i, \\ 0 & \text{else.} \end{cases}$$

For $i < 0$, X has a nontrivial negative homotopy group.

Spectra that have their homotopy groups concentrated in nonnegative degrees are particularly nice, and are called *connective*. If there is some N so that $\pi_k(X) = 0$ for all $k < N$, we say the spectrum is *bounded below*.

We see below that having an Eilenberg-MacLane spectrum is very fruitful. In particular, we can define a homology and cohomology theory. We will use these theorems later in computations in Chapter 4.

Theorem 2.2.22 (Brown Representability, [19, Thm. 4E.1/Thm. 4F.2])**.** *Every (reduced) cohomology theory on based CW complexes has the form* $h^n(X) = [X, K_n]$ *for some sequential spectrum* K *given by* $\{K_n\}$*. The groups* $h_i(X) = \pi_i(X \wedge K)$ *form a reduced homology theory.*

Theorem 2.2.23 ([19, Prop. 4F.1])**.** *For* X *a based CW complex, stable homotopy groups* $\pi_n^s(X)$ *define a (reduced) homology theory.*

Example 2.2.24. Let G be an abelian group. With these theorems, HG is the representing object for $H^*(-;G)$ and $H_*(-;G)$. We can think of (co)homology with coefficients in G as HG (co)homology. In particular, we have $HG^n(X) = [X, \Sigma^n HG]$ and $HG_n(X) = \pi_n(X \wedge HG)$.

2.3 Categorical Detour

It is convenient to have many categorical notions only up to homotopy. We record a few standard definitions and models. Throughout this section we work with spectra. In the stable category of spectra there is a *zero object*, denoted by 0, which is both initial and terminal, so mapping cones/fibers may be expressed using 0.

Definition 2.3.1. A map $f: X \to Y$ of spectra is a *(stable) fibration* if each $f_n: X_n \to Y_n$ is a fibration of spaces and the natural map $X_n \to Y_n \times_{\Omega Y_{n+1}} \Omega X_{n+1}$ is a weak equivalence of spaces. A map $f: X \to Y$ of spectra is a *(stable) cofibration* if each $f_n: X_n \to Y_n$ is a cofibration of spaces and the induced "latching" maps $X_n \cup_{\Sigma Y_{n-1}} \Sigma X_{n-1} \to X_n$ are cofibrations.

Definition 2.3.2. A spectrum X is *cofibrant* if the unique map from the initial object $0 \to X$ is a cofibration. A spectrum Y is *fibrant* if the unique map to the terminal object $Y \to 0$ is a fibration.

We think of cofibrant as a good source and fibrant as a good target.

Definition 2.3.3. The *homotopy colimit* of a diagram $F: I \to \mathcal{C}$ is given by replacing F by a cofibrant diagram, then taking the colimit. The homotopy limit of a diagram $F: I \to \mathcal{C}$ is dual. It is given by replacing F by a fibrant diagram, then taking the limit.

Homotopy (co)limits are homotopy invariant (co)limits.

Definition 2.3.4. Given a map of spectra $f\colon X \to Y$, the *homotopy cofiber* is $\mathrm{hocofib}(f) \coloneqq \mathrm{hocolim}(X \xrightarrow{f} Y \leftarrow 0)$. It fits into a triangle

$$X \xrightarrow{f} Y \to \mathrm{hocofib}(f) \to \Sigma X.$$

The *homotopy fiber* is $\mathrm{hofib}(f) \coloneqq \mathrm{holim}(X \xrightarrow{f} Y \leftarrow 0)$. It fits into a triangle

$$\Omega Y \to \mathrm{hofib}(f) \to X \xrightarrow{f} Y.$$

We think of a homotopy cofiber as the spectrum version of a mapping cone and homotopy fiber as the spectrum version of a mapping fiber.

Definition 2.3.5. Given maps $f, g\colon X \to Y$, the *homotopy equaliser* is

$$\begin{aligned} \mathrm{hoeq}(f,g) \coloneqq & \varprojlim_{\mathrm{holim}} \left(X \underset{g}{\overset{f}{\rightrightarrows}} Y \right) \\ =& \{(x,\gamma) \mid x \in X, \gamma\colon I \to Y, \gamma(0) = f(x), \gamma(1) = g(x)\}. \end{aligned}$$

This enforces equality $f \simeq g$ up to homotopy. The *homotopy coequaliser* is

$$\begin{aligned} \mathrm{hocoeq}(f,g) \coloneqq & \underrightarrow{\mathrm{hocolim}} \left(X \underset{g}{\overset{f}{\rightrightarrows}} Y \right) \\ =& (Y \sqcup (X \times I))/\{(x,0) \sim f(x), (x,1) \sim g(x)\}. \end{aligned}$$

This identifies f with g homotopically.

With this language, we can give a few more important examples of spectra.

Example 2.3.6. Let p be a prime and let R be a ring spectrum. The *mod-p spectrum* (sometimes Moore spectrum) R/p is defined as the homotopy cofiber of multiplication by p on R, induced by levelwise degree p maps:

$$R \xrightarrow{p} R \to R/p.$$

The *p-complete spectrum* $R_p^\wedge$ is given by the homotopy limit

$$R_p^\wedge \coloneqq \underset{n}{\underleftarrow{\mathrm{holim}}}(\cdots \to R/p^n \to R/p^{n-1} \to \cdots).$$

The *p-local spectrum* $R_{(p)}$ is given by inverting all primes away from p. That is,

$$R_{(p)} = \underrightarrow{\mathrm{hocolim}}(R \xrightarrow{p_1} R \xrightarrow{p_1 p_2} R \to ...)$$

for $p_i \neq p$ and the p_i in order.

Chapter 3

Equivariant Background

We begin the chapter with the basic toolkit and language for equivariant homotopy theory. From these foundations, we expand on Chapter 2 and discuss equivariant stable homotopy theory. The equivariant Whitehead theorem 3.1.9 provides a first indication that equivariant homotopy is the right setting: it shows that in order to recover a homotopical classification analogous to the nonequivariant case, we must keep track of fixed point information and how these interact across different subgroups.

Next we will see how algebraic structures such as Mackey functors capture equivariant homotopy groups and how additional multiplicative structure leads to Green functors and Tambara functors.

Finally we introduce two key constructions that play a central role in modern equivariant homotopy theory: the Tate construction which interpolates between fixed points and orbits, and the norm functor, which encodes multiplicative transfers. These tools provide the technical foundation for Chapters 4 and 5, particularly in the study of topological Hochschild homology and later twisted topological Hochschild homology.

In this chapter, we allow G to be a compact Lie group or a finite group. While many statements hold in the more general compact Lie case, for concreteness and simplicity, we often restrict to finite groups or abelian compact Lie groups.

3.1 Equivariant Homotopy Theory

In this section, we provide some of the basic definitions, theorems and examples that we will need from the ordinary equivariant homotopy theory of spaces, before moving to spectra.

We start by giving a few examples of basic G-spaces.

Example 3.1.1. For any space X and any topological group G, X is a G-space with the trivial G-action $G \times X \to X$ given by $(g, x) \mapsto x$ for all $g \in G$.

Examples 3.1.2. Let $G = C_2 = \{e, \gamma\}$, then S^2 is a C_2-space with the antipodal action $\gamma x = -x$. We will write S^n_{anti} for S^n with the antipodal action. This action is also free since $0 \notin S^2_{\text{anti}}$, so there is no $x \in S^2$ such that $x = -x$. In particular, S^n_{anti} is not a *based* C_2-space.

We can see that S^2 is also a C_2-space with action given by reflection in the x_0x_1-plane: $\gamma(x_0, x_1, x_2) = (x_0, x_1, -x_2)$. We will write S^n_{ref} for S^n with the reflection action. The fixed point set $(S^2_{\text{ref}})^{C_2} = \{(x_0, x_1, 0) \in S^2\} \cong S^1$ is the equator.

We can see that S^2 is also a C_2-space with action given by rotation in the axis through the basepoint. We will write S^n_{rot} for S^n with the rotation action. The fixed point set $(S^2_{\text{rot}})^{C_2} \cong S^0$ is the two poles.

These examples show that even with the same group and space, there are different ways of (nontrivially) turning the space into a G-space.

Example 3.1.3. Let G be a compact Lie group. Let V be a finite dimensional real representation of G with $\dim V = n$. The *representation sphere* $S^V := V \cup \{\infty\}$ is the one point compactification of V. Then S^V is a based G-space with trivial action on the base point. If $V = \mathbb{R}^n$ with the trivial G-action, then $S^V = S^n$ with the trivial G-action. So representation spheres are a generalisation of the usual spheres S^n.

When we move to equivariant stable homotopy theory, representation spheres play the role of spheres in the nonequivariant theory.

We are interested in studying homotopy classes of equivariant maps between G-spaces.

Definition 3.1.4. Let X and Y be G-spaces and $f, g : X \to Y$ be G-maps. An *equivariant homotopy* between f and g is a G-equivariant map $F\colon X \times I \to Y$, where I is given the trivial action, such that $F|_{X\times\{0\}} = f$ and $F|_{X\times\{1\}} = g$. If $H \subset G$ is a subgroup, we write homotopy classes of H-maps between X and Y as $[X, Y]^H$.

We want to translate the notion of a CW complex to the equivariant world. This is useful in many proofs, being able to induct over skeleta and getting easy filtrations for spectral sequences. The naive way to define a G-CW complex would be a CW complex with a G-action. However, with this definition we would not recover a Whitehead theorem in the equivariant setting as in Theorem 3.1.9. We explain this alongside the theorem. Instead, we impose a lot more equivariant structure on each cell, as we will see is the theme throughout.

Definition 3.1.5 ([13, §1.1])**.** A *G-CW complex* is a G-space X which is given by a sequence of based G-spaces X^n, each obtained from X^{n-1} by attaching a disjoint union of G-spaces $G/H_i \times D^n$ along the boundary $G/H_i \times S^{n-1}$ by an equivariant map, for some set of closed subgroups $\{H_i\}$ of G.

Note in particular that for $H \subset G$ closed, $G/H \times D^n$ is a single n-cell.

Example 3.1.6. Following on from Examples 3.1.2, S^2_{anti} is a C_2-CW complex with one free 0-cell $C_2/e \times D^0$, one free 1-cell $C_2/e \times D^1$ and one free 2-cell $C_2/e \times D^2$. Note that nonequivariantly, the resulting space is S^2, and every attachment map is C_2-equivariant since we always attach whole orbits at a time, and the antipodal action swaps the corresponding pieces. We also note that the orbit space after each stage is the nonequivariant CW structure of $\mathbb{R}P^2$.

Definition 3.1.7. Let (X, x_0) be a based G-space. The *equivariant homotopy groups* of X for closed $H \subset G$, at $x_0 \in X^H$ are $\pi_n^H(X, x_0) \coloneqq \pi_n(X^H, x_0)$, for all $n \in \mathbb{N}$. Here we write $X^H = \{x \in X \mid Hx = x\}$ for the H-fixed points, with the subset topology.

As in the nonequivariant case, if there is a canonical basepoint in X or it is clear from context, it is common to suppress the notation to just $\pi_n^H(X)$.

Example 3.1.8. Let $G = C_2$, $X = S^2_{\text{ref}}$. We know that $(S^2_{\text{ref}})^{C_2} \cong S^1$, so we have

$$\pi_n^e(S^2_{\text{ref}}) = \pi_n(S^2_{\text{ref}}) = \pi_n(S^2)$$

$$\pi_n^{C_2}(S^2_{\text{ref}}) = \pi_n((S^2_{\text{ref}})^{C_2}) \cong \pi_n(S^1) \neq 0, \quad \text{for } n = 1.$$

For $n = 1$, this is $\mathbb{Z}$. In particular, the extra $\mathbb{Z}$ in $\pi_1(S^1)$ detected at the fixed points is invisible to nonequivariant homotopy. So, by looking at this reflection action on S^2 and computing the equivariant homotopy groups, we have revealed more information than nonequivariant homotopy groups can show.

With these definitions, we use the following theorem as further motivation for studying equivariant homotopy theory.

Theorem 3.1.9 (Equivariant Whitehead)**.** *A based map $f : X \to Y$ of based G-CW complexes is a G-equivalence (G-equivariant homotopy equivalence) if and only if f induces isomorphisms*

$$f_*^H : \pi_k^H(X, x_0) \xrightarrow{\cong} \pi_k^H(Y, f(x_0))$$

for all closed $H \subset G$, for all $k \in \mathbb{N}$, and for all $x_0 \in X^H$.

Proof. If f is a G–homotopy equivalence, then taking H–fixed points preserves homotopies and gives a homotopy equivalence f^H, hence a weak equivalence, for all H. For the other direction, we can assume that f is G-homotopic to a G-CW map (one that maps n-skeleton to n-skeleton) using CW approximation [30, I.2.6]. We construct a G–map $F\colon Y \to X$ and G–homotopies $f \circ F \simeq_G \mathrm{id}_Y$ and $F \circ f \simeq_G \mathrm{id}_X$ by induction on skeleta.

Suppose we have G–maps $F_n\colon Y^n \to X^n$ and a G–homotopy $\Phi_n\colon f \circ F_n \simeq_G \mathrm{id}_{Y^n}$. Let an $(n+1)$–cell of Y have orbit type G/H, attached via a G–map $\alpha\colon G/H \times S^n \to Y^n$. To extend F_n across this cell, we need the composite

$$G/H \times S^n \xrightarrow{\alpha} Y^{(n)} \xrightarrow{F_n} X^{(n)} \hookrightarrow X$$

to be G–nullhomotopic. With this in mind, we consider the "obstruction class" $[\omega]$ in the equivariant homotopy set

$$[G/H_+ \wedge S^n, X]^G \cong [S^n,\ X^H] \cong \pi_n(X^H),$$

using the adjunction $[G/H_+ \wedge A, Z]^G \cong [A, Z^H]$. Under f, this obstruction maps to the obstruction to extending $\mathrm{id}_{Y^{(n)}}$ across the cell, which is zero. Since $f_*^H : \pi_n(X^H) \to \pi_n(Y^H)$ is an isomorphism by hypothesis, the original obstruction must also vanish. Hence F_n extends to $F_{n+1}\colon Y^{(n+1)} \to X^{(n+1)}$.

A relative version of the same argument extends the homotopy Φ_n to a G–homotopy $\Phi_{n+1}\colon f \circ F_{n+1} \simeq_G \mathrm{id}_{Y^{n+1}}$. Passing to the colimit over n yields a G–map $F\colon Y \to X$ with $f \circ F \simeq_G \mathrm{id}_Y$. Repeating the construction with X, Y interchanged, we obtain a G–homotopy $\Psi\colon F \circ f \simeq_G \mathrm{id}_X$. Thus f is a G–homotopy equivalence. □

For this theorem and proof, we needed G-CW complexes as defined in Definition 3.1.5. If instead we defined a G-CW complex to be a CW-complex with a G-action there can be an n-cell which has points with stabilisers in different conjugacy classes.

Example 3.1.10. Let $G = C_2$ and consider D^2_{rot} the disk D^2 with rotation action. Think of D^2 as a single 2-cell in a naive G-CW sense. Then $(D^2_{\text{rot}})^{C_2} \cong *$ the center and $(\partial D^2_{\text{rot}})^{C_2} = (S^1_{\text{rot}})^{C_2} = \emptyset$. Thus, the pair $((D^2)^{C_2}, (S^1)^{C_2}) = (*, \emptyset)$. After taking fixed points, we no longer have a disk with a sphere boundary in the right degree.

This example exhibits what goes wrong with the naive definition of G-CW complexes.

3.2 Equivariant Stable Homotopy Theory

Spectra were the correct objects to use in the case of nonequivariant stable homotopy theory, so we want some notion of equivariant spectra. There are a couple of ways that one could come up with such a category, and we discuss this. As in Chapter 2, in what follows, when we say spectrum, we mean orthogonal spectrum.

A key feature of the equivariant theory is that suspension is done not only by spheres S^n but by representation spheres S^V. To carry this out in a systematic way, we introduce the notion of a G-universe:

Definition 3.2.1. A *G-universe* $\mathcal{U}$ is an infinite dimensional real inner product space with a linear G-action such that $\mathcal{U}$ contains the trivial G-representation and if $V \subset \mathcal{U}$ is a finite dimensional orthogonal G-representation, then countably infinitely many pairwise orthogonal copies of V are also contained in $\mathcal{U}$.

Example 3.2.2. The trivial G-universe is $\mathcal{U}_{\text{triv}} = \bigoplus_{n\in\mathbb{N}} \mathbb{R} \cong \mathbb{R}^\infty$ with G acting trivially on each summand.

A G-universe is *complete* if every finite dimensional irreducible orthogonal G-representation occurs in $\mathcal{U}$ with infinite multiplicity. When doing equivariant stable homotopy theory, stabilisation is done with respect to a fixed complete G-universe. That is, suspension is allowed by every S^V and desuspension by S^V is also invertible. Note that any two complete G-universes are isomorphic, so the resulting homotopy theory is independent of $\mathcal{U}$ [37, §5]. We will fix $\mathcal{U}$ as our complete universe.

Definition 3.2.3. A *naive G-spectrum* X is a spectrum (in the nonequivariant sense) with each X_n a based G-space, and each σ_n being G-equivariant.

In the language of universes, a naive G-spectrum is a G-spectrum indexed on the trivial universe. Naive G-spectra are too restrictive, while they allow one to speak of suspension by the ordinary spheres S^n, they do not interact correctly with representation spheres and fixed-points. For example, fixed points of naive suspension spectra fail to recover the expected homotopy types (see [22, Ex. 9.3.11]). We also see this in Example 3.2.7.

Definition 3.2.4. A *genuine G-spectrum* X is a spectrum indexed on the complete G–universe $\mathcal{U}$. For $V \subset \mathcal{U}$, we will write $\Sigma^V X$ for $X \wedge S^V$ with the diagonal G-action. In particular, X is built out of G-spaces indexed by finite dimensional G-representations $V \subset \mathcal{U}$, $\{X(V)\}_{V \subset \mathcal{U}}$, together with G-maps $\sigma\colon \Sigma^W X(V) \to X(V \oplus W)$.

We give one of the easiest nontrivial examples of a genuine G-spectrum.

Example 3.2.5. Let $G = C_2$ and consider $\mathbb{S}$ with the trivial C_2-action, denoted $\mathbb{S}_{C_2}$. For each finite dimensional C_2-representation V, define $\mathbb{S}_{C_2}(V) = S^V$, a based C_2-space. The structure maps are given by the canonical homeomorphisms $\sigma\colon S^V \wedge S^W \xrightarrow{\cong} S^{V \oplus W}$. This assembles all spheres S^V into a genuine C_2-spectrum.

Another reason for introducing universes is to define homotopy groups systematically.

Definition 3.2.6. Let X be a genuine G-spectrum and let $H \subset G$ be a closed subgroup. The *equivariant homotopy groups* of X graded on $\mathcal{U}$ are

$$\pi_V^H(X) := \varinjlim_{W \subset \mathcal{U}|_H} [S^{V \oplus W}, X(W)]^H,$$

where the colimit runs over the finite-dimensional H–subrepresentations W of the restricted universe.

With this definition, note that $\Sigma^V(-)$ from Definition 3.2.4 shifts the homotopy groups of a genuine G-spectrum up by $[V]$.

We can see that the equivariant homotopy groups of genuine G-spectra hold more information than the nonequivariant or naive G-spectra counterparts.

Example 3.2.7. Let σ be the sign representation for C_2. Consider $\mathbb{S}_{C_2}$, $S^0_{\text{triv}} = \{0, \infty\}$ and the C_2-inclusion $\iota\colon S^0 \to S^\sigma$. Suspending ι gives maps $\Sigma^n \iota\colon S^n \to S^{n+\sigma}$. This gives an element in $\pi_{-\sigma}^{C_2}(\mathbb{S}_{C_2}) \cong \mathbb{Z}$. In fact, this is the first Euler/orientation class found in the slice spectral sequence in [22].

Note that, as a naive C_2-spectrum, $\mathbb{S}$ only admits suspension by S^n, not by S^σ. As a result, the homotopy groups would not capture the Euler class found in this previous example.

A map of orthogonal G-spectra is a *stable equivalence* if it induces isomorphisms $\pi_V^H(X) \xrightarrow{\cong} \pi_V^H(Y)$ for all closed subgroups $H \subset G$ and all $V \subset \mathcal{U}$. The equivariant stable homotopy theory is done in a category that is obtained from the category of orthogonal G-spectra by formally inverting these stable equivalences. In this category, suspension by any finite dimensional G-representation V becomes an isomorphism.

Theorem 3.2.8 ([22, Thm. 9.1.25])**.** *The category of (orthogonal) genuine G-spectra Sp^G is a closed symmetric monoidal category with unit $\mathbb{S}_G$. The right adjoint to the smash product with Y is the internal Hom functor $F_G(Y,-)$. The category is also enriched, tensored and cotensored over based G-spaces.*

From this point onward, we will only say G-spectrum, when we mean genuine G-spectrum, unless specifically said otherwise.

Definition 3.2.9. Let A be a based G-space and let X be a G-spectrum. Define the *tensor* $X \wedge A$ to be the G-spectrum with V-th space $(X \wedge A)(V) = X(V) \wedge A$. Define the *cotensor* $M(A, X)$ to be the G-spectrum with V-th space $M(A,X)(V) = \mathrm{Map}_*(A, X(V))$.

The cotensor is sometimes written as $F(A,-)$, but we use $M(A,-)$ to avoid confusion with the internal Hom functor $F_G(-,-)$. Note that in the definition, $M(A,X)(V)$ is the space of based maps from $A \to X(V)$, and is *not* the space of based G-maps from $A \to X(V)$. For the latter, we will denote this by $M^G(A,X)$.

We go through exactly how each space in these spectra are G-spaces. First, we can see that each space $(X \wedge A)(V) = X(V) \wedge A$ is a G-space with diagonal G-action. We also see that $M(A,X)(V) = \mathrm{Map}_*(A, X(V))$ is a G-space with conjugation action $(g \cdot f)(a) = g(f(g^{-1}a))$. If we take categorical G-fixed points of this space, we instead recover all based G-maps,

$$\begin{aligned} M^G(A,X) = \mathrm{Map}_*(A, X(V))^G &= \{f \colon A \to X(V) \mid g(f(g^{-1}a)) = f(a)\} \\ &= \{f \colon A \to X(V) \mid f \text{ is } G\text{-equivariant}\}. \end{aligned}$$

This justifies our notation. With all this, Theorem 3.2.8 gives that for a based G-space A and genuine G-spectrum X, both $X \wedge A, M(A,X) \in \mathrm{Sp}^G$.

It turns out that S^0 is both the unit of the tensor and of the cotensor.

Theorem 3.2.10. *Let X be a G-spectrum and give S^0 the trivial G-action, then $M(S^0, X) \cong X$.*

Proof. We know that $M(S^0, X)(V) = \mathrm{Map}_*(S^0, X(V))$. We can see that this space of maps is $X(V)$ since the basepoint of S^0 maps to the basepoint of $X(V)$ and the other point in S^0 maps to anywhere in $X(V)$, recovering the whole space. These levelwise identifications are natural in V, so glue together to give an isomorphism of spectra. □

Definition 3.2.11. Let X be a genuine G-spectrum. The *V-th loops* of X is $\Omega^V X := M(S^V, X)$, with the conjugation G-action.

As in the nonequivariant case, there is an adjunction between Σ^V and Ω^V.

Theorem 3.2.12. *For any $V \subset \mathcal{U}$, there is an adjunction*

$$\Sigma^V : \mathrm{Sp}^G \rightleftarrows \mathrm{Sp}^G : \Omega^V.$$

Proof. The category Sp^G has smash product and function spectrum which are adjoint, and these are how we defined Σ^V and Ω^V. This follows from the closed symmetric monoidal structure of Sp^G. □

Similar to G-spaces, given a G-spectrum, we can construct another equivariant spectrum by taking fixed points:

Definition 3.2.13. Let X be a G-spectrum. For each closed subgroup $H \subset G$ we have a *(categorical) H-fixed point spectrum* X^H given by $X^H(V) = (X(V))^H$, where V is a G-representation fixed by H.

We soon state Theorem 3.2.16 as a way to compute these fixed point spectra. In order make sense of the statement of the theorem, we introduce some definitions.

Definition 3.2.14. Let G be a compact Lie group. The *Weyl group* of $H \subset G$ is the quotient

$$W_G H = N_G H/H = \{g \in G \mid gH = Hg\}/H.$$

We see for G also abelian that $W_G H = G/H$.

For $H \subset G$ closed, if X is a G-spectrum, then X^H is $W_G H$-spectrum [28, Ch V Prop. 3.4]. This underpins why the Weyl group appears when talking about categorical fixed points.

Definition 3.2.15. A *universal G-space EG* is a G-space with free G-action and the underlying topological space of EG is contractible. This is unique up to G-homotopy equivalence.

The quotient EG/G recovers the classifying space BG. The quotient map $\pi\colon EG \to BG$ is a principal G-bundle, and is called the *universal principal G-bundle*. For any principal G-bundle $E \to X$, there exists a map $f\colon X \to BG$ which is unique up to homotopy, such that $E \cong f^*(EG)$.

Theorem 3.2.16 (tom Dieck Splitting [40, Thm. 6.12])**.** *Let G be a finite group. There is a natural weak equivalence of (nonequivariant) spectra*

$$(\Sigma^\infty X)^G \simeq \bigvee_{(H)\subset G} \Sigma^\infty \left((EW_G H)_+ \wedge_{W_G H} X^H\right)$$

where the wedge runs over conjugacy classes (H) of subgroups $H \subset G$.

Consequently,

$$\pi_k\left((\Sigma^\infty X)^G\right) \cong \bigoplus_{(H)\subset G} \pi_k\left(\Sigma^\infty\left((EW_G H)_+ \wedge_{W_G H} X^H\right)\right).$$

The tom Dieck splitting holds more generally for any compact Lie group. The idea of the proof in this case is similar, but one needs to be careful with possibly infinite wedge sums and direct sums.

If G is finite and abelian, then the formula simplifies to

$$(\Sigma^\infty X)^G \simeq \bigvee_{H\subset G} \Sigma^\infty\left((E(G/H))_+ \wedge_{G/H} X^H\right).$$

Categorical fixed points see a direct summand from every subgroup of G, so they cannot easily isolate information intrinsic to the whole group. In §3.3 we look at other types of fixed points that work nicer with G-spectra.

3.3 The Tate Construction

The Tate construction is required to define important spectra in equivariant homotopy theory. These spectra allow us to define cyclotomic spectra and Tate cohomology, which are required in later chapters. Throughout this section, let $G = C_p$ and let X be a C_p-spectrum.

Denote by $\widetilde{EG}$ the reduced mapping cone of the based G-map $EG_+ \to S^0$ that sends EG to the nonbasepoint of S^0. This means that $\widetilde{EG}$ is the unreduced suspension of EG. Now, we consider this cofiber sequence $EG_+ \to S^0 \to \widetilde{EG}$. Note that nonequivariantly $EG_+ \simeq S^0$, so $\widetilde{EG} \simeq *$, and equivariantly $(EG_+)^G \simeq *$, so $\left(\widetilde{EG}\right)^G \simeq S^0$.

Smashing with a G-spectrum X preserves cofiber sequences [29, §II, III], so we get

$$EG_+ \wedge X \to X \to \widetilde{EG} \wedge X.$$

Recall from Definition 3.2.9 that for a X a based G-space, $M(A, X)$ is a spectrum with V-th space $M(A, X)(V) = \mathrm{Map}_*(A, X(V))$. With this in mind, we can do the construction as before with $M(EG_+, X)$ in place of X and get

$$EG_+ \wedge M(EG_+, X) \to M(EG_+, X) \to \widetilde{EG} \wedge M(EG_+, X).$$

To bring these two sequences together into a single diagram, we construct a map $X \to M(EG_+, X)$. First, write $S^0 = \{*, 1\}$. Then there is a unique based G-map $\varepsilon\colon EG_+ \xrightarrow{\simeq} S^0$ given by $* \mapsto *$, $EG \mapsto 1$. Since every point of EG goes to the same nonbasepoint, ε is G-equivariant. Given a based G-map $f\colon A \to B$, and a G-spectrum X, we get a morphism of spectra $M(B, X) \xrightarrow{M(f,X)} M(A, X)$ given by precomposing with f. Applying this to $f = \varepsilon$, and using Theorem 3.2.10, we get a map $\theta = M(\varepsilon, X)\colon X \cong M(S^0, X) \to M(EG_+, X)$.

With this map, we now have the commutative diagram

$$\begin{array}{ccccc}
EG_+ \wedge X & \longrightarrow & X & \longrightarrow & \widetilde{EG} \wedge X \\
\downarrow & & \downarrow{\scriptstyle\theta} & & \downarrow \\
EG_+ \wedge M(EG_+, X) & \longrightarrow & M(EG_+, X) & \longrightarrow & \widetilde{EG} \wedge M(EG_+, X).
\end{array}$$

Finally, we take categorical G fixed points of this diagram,

$$\begin{array}{ccccc}
(EG_+ \wedge X)^G & \longrightarrow & X^G & \longrightarrow & \left(\widetilde{EG} \wedge X\right)^G \\
{\scriptstyle\simeq}\downarrow & & \downarrow & & \downarrow \\
(EG_+ \wedge M(EG_+, X))^G & \longrightarrow & (M(EG_+, X))^G & \longrightarrow & \left(\widetilde{EG} \wedge M(EG_+, X)\right)^G.
\end{array}$$

The weak equivalence in the diagram can be seen in [18].

With all of this setup, we introduce notation for all of the spectra involved in the diagram below, as they play a large role in equivariant homotopy theory.

$$\begin{array}{ccccc}
X_{hG} & \longrightarrow & X^G & \longrightarrow & \Phi^G X \\
{\scriptstyle\simeq}\downarrow & & \downarrow & & \downarrow \\
X_{hG} & \xrightarrow{S} & X^{hG} & \longrightarrow & X^{tG}
\end{array}$$

In the literature, the map $X_{hG} \to X^{hG}$ is usually called the norm map, denoted Nm. We instead use S for this, and call it the splice map, to avoid overuse of maps being called a norm.

Definition 3.3.1. Let $G = C_p$ and let X be a G-spectrum,

- the *homotopy orbit spectrum* of X is $X_{hG} \coloneqq (EG_+ \wedge X)^G$,
- the *homotopy fixed point spectrum* of X is $X^{hG} \coloneqq (M(EG_+, X))^G$,
- the *geometric fixed point spectrum* of X is $\Phi^G X \coloneqq \left(\widetilde{EG} \wedge X\right)^G$,
- the *Tate spectrum* of X is $X^{tG} \coloneqq \text{hocofib}\left(X_{hG} \xrightarrow{S} X^{hG}\right)$.

The below lemma identifies X^{tG} with the spectrum from the previous diagram.

Lemma 3.3.2 ([34, Prop. II.2.13])**.** *Let X be a G-spectrum. Then there is an equivalent description of the Tate spectrum $X^{tG} \simeq \left(\widetilde{EG} \wedge M(EG_+, X)\right)^G$.*

For groups $G \neq C_p$, some of the definitions are slightly more complicated and involve looking at all of the proper subgroups of G, which is why C_p was easy. In the more general case, the formulation is extremely similar, but there is some more going on in the background, see [17].

Along the top of the diagram, we find the so-called *isotropy separation* sequence which splits the categorical fixed points of X into the homotopy orbits part and the geometric fixed points part.

The Tate construction is not only useful for the Tate spectrum, but also for the other spectra in Definition 3.3.1. The categorical fixed points used for homotopy groups are the most immediate form of fixed points, though this notion of fixed points has some interesting and unexpected properties. With categorical fixed points, recall that if X is a G-spectrum, then X^H is a $W_G H$-spectrum, but note that $(X \wedge Y)^H \not\simeq X^H \wedge Y^H$ and $(\Sigma^\infty X)^H \not\simeq \Sigma^\infty(X^H)$.

Example 3.3.3. We use the tom Dieck splitting to show this. Take $G = C_2$ and $X = S^0$, so

$$W_G H = \begin{cases} e & H = C_2 \\ C_2 & H = e. \end{cases}$$

By the tom Dieck splitting we find that,

$$\begin{aligned}(\Sigma^\infty S^0)^{C_2} &\simeq \Sigma^\infty((EC_2)_+ \wedge_{C_2} S^0) \vee \Sigma^\infty((Ee)_+ \wedge_e S^0) \\ &\simeq \Sigma^\infty \mathbb{R}P^\infty_+ \vee \mathbb{S}.\end{aligned}$$

In particular, note that this is not the expected $\Sigma^\infty\left((S^0)^{C_2}\right) \simeq \mathbb{S}$. Hence, $(\Sigma^\infty X)^H \not\simeq \Sigma^\infty(X^H)$.

Instead, we consider geometric fixed points $\Phi^G X$, which have the properties that we originally expected from fixed points, such as $\Phi^G(\Sigma_G^\infty X) \cong \Sigma^\infty(X^G)$. Geometric fixed points commute with suspension by representation spheres and smash products with nonequivariant spectra, while homotopy fixed points and orbits have descent and duality formalisms useful for spectral sequences (as we will see in Theorem 4.4.3). We can use geometric fixed points and the Tate spectrum to define cyclotomic spectra. This cyclotomic structure plays an important role throughout the later chapters, in particular we will see that topological Hochschild homology is cyclotomic in Chapter 4, and this is what gives us topological cyclic homology, defined in Definitions 4.4.1 and 4.4.2. The old definition of a cyclotomic spectrum is given below.

Definition 3.3.4 ([6, Def. 4.9, 4.10])**.** A *cyclotomic spectrum* is a genuine S^1-spectrum X together with compatible equivalences $\Phi^{C_n} X \xrightarrow{\simeq} X$ for all $n \in \mathbb{N}$. Note that $\Phi^{C_n} X$ has a residual S^1/C_n-action, which we identify with an S^1-action through the n-th power map $S^1/C_n \xrightarrow{\cong} S^1$.

This classical definition requires a lot of computation to determine whether a spectrum is cyclotomic. Due to work of Nikolaus and Scholze [34], we can define a cyclotomic spectrum in terms of the Tate construction instead of the classical Definition 3.3.4.

Definition 3.3.5 ([34, Def. II.1.1.])**.**

1. A *cyclotomic spectrum* is a genuine S^1-spectrum X together with S^1-maps $\varphi_p \colon X \to X^{tC_p}$ for every prime p. Similar to before, note that X^{tC_p} has a residual S^1/C_p-action, which we identify with an S^1-action through the p-th power map $S^1/C_p \xrightarrow{\cong} S^1$.

2. For a fixed prime p, a *p-cyclotomic spectrum* is a genuine C_{p^∞}-spectrum with a C_{p^∞}-map $\varphi_p \colon X \to X^{tC_p}$. Here, $C_{p^\infty} \subset S^1$ is the subgroup of p-power torsion, and we identify the C_{p^∞}-action on X^{tC_p} with $C_{p^\infty}/C_p \cong C_{p^\infty}$ through the p-th power map. Here we think of $C_{p^\infty} \subset S^1$ as the "p-part" of S^1.

This new definition of cyclotomic spectra is much easier to work with as we do not need to check compatibility conditions of the maps. The two definitions are not literally identical. However, Nikolaus and Scholze prove that they agree for bounded below spectra, which is the case of interest for our computations, including THH. Since we only care about the connective setting, the distinction will not affect our applications. Further still, p-cyclotomic will suffice for us to define $\mathrm{TC}(-;p)$ [34, Rmk. II.1.3], and often working with C_{p^∞} can be simpler than S^1.

To give a simple example of a cyclotomic spectrum, we first need to make a note about how to check whether a map is G-equivariant. Let X, Y be G-spectra where X has the trivial G-action, then we have $F_G(X,Y) \cong F(X, Y^{hG})$. That is, G-maps out of X are the same as any maps $X \to Y^{hG}$. To see this, note that since X has trivial action, a G-map $X \to Y$ is the same as a map $X \to M^G(EG_+, X)$. This is the spectra analogue of the fact for G-spaces that a G-map $X \to Y$ is the same as a map $X \to Y^G$.

Example 3.3.6 ([34, Ex. II.1.2])**.** Consider $\mathbb{S}$ with the trivial S^1-action. There are maps $\varphi_p \colon \mathbb{S} \to \mathbb{S}^{tC_p}$ given by $\mathbb{S} \to \mathbb{S}^{hC_p} \to \mathbb{S}^{tC_p}$. To make these maps equivariant it suffices to construct a map factoring through $\mathbb{S}^{hS^1}$ to make sure the Frobenius is an S^1-map. So, we lift the map $\mathbb{S} \to \mathbb{S}^{tC_p}$ to a map $\mathbb{S} \to \left(\mathbb{S}^{tC_p}\right)^{h(S^1/C_p)}$. This suffices since the S^1-action on $\mathbb{S}$ is trivial. We have maps

$$\mathbb{S} \to \mathbb{S}^{hS^1} \simeq \left(\mathbb{S}^{hC_p}\right)^{h(S^1/C_p)} \to \left(\mathbb{S}^{tC_p}\right)^{h(S^1/C_p)}.$$

This shows that $\mathbb{S}$ is cyclotomic.

In Theorem 4.2.15, we will see another crucial example of a cyclotomic spectrum.

Example 3.3.7 ([34, Ex. II.1.2])**.** For every cyclotomic spectrum, we get a p-cyclotomic spectrum by restriction.

In practice, we will work with the p-completion of objects, so the p-cyclotomic structure will already suffice for our purposes.

3.3.1 Tate cohomology

We can also use this Tate construction for the *Tate cohomology* of a compact Lie group.

Definition 3.3.8. Let G be a group, and let M be a G-module. The *group cohomology* of G with coefficients in M is $H^n_{\mathrm{grp}}(G;M) := H^n(BG;M)$. Similarly, the *group homology* of G with coefficients in M is $H_n^{\mathrm{grp}}(G;M) := H_n(BG;M)$.

Tate cohomology splices group homology and cohomology together into one:

Definition 3.3.9. Let G be a finite group and M a G-module. The *Tate cohomology* of G with coefficients in M, $\widehat{H}^n_{\text{grp}}(G;M)$, is obtained by splicing a projective resolution with its dual to form a complete resolution. Equivalently, one can use the standard description

$$\widehat{H}^n_{\text{grp}}(G;M) = \begin{cases} H^n_{\text{grp}}(G;M) & n \geq 1, \\ \ker\big(S\colon M_G \to M^G\big) & n = -1, \\ \operatorname{coker}\big(S\colon M_G \to M^G\big) & n = 0, \\ H^{\text{grp}}_{1-n}(G;M), & n \leq -2, \end{cases}$$

where $M_G \coloneqq M/\langle gm - m\rangle$, and $S\colon M_G \longrightarrow M^G$ given by $[m] \longmapsto \sum_{g\in G} gm$ is the *splice map* (as seen before). Again, the literature conventionally calls this the norm map, but to avoid conflicting use of this term, we call it the splice map.

We quickly show that the splice map is well-defined. If $[m] = [m']$ with $m' = m + (gx - x)$, then

$$S(m') - S(m) = \sum_{h\in G} h(gx - x) = \sum_{h\in G} hgx - \sum_{h\in G} hx = 0$$

since $\{hg \mid h \in G\} = \{h \mid h \in G\}$ as G is finite. Hence, this is well-defined without reference to the order of summation.

Note that some authors shift indices by 1 in Tate cohomology, but our convention is compatible with the Tate spectral sequences used later.

Example 3.3.10. Let $G = C_p = \langle\gamma\rangle$ and $M = \mathbb{F}_p$ with the trivial action. Consider the 2-periodic projective resolution over $\mathbb{Z}[C_p]$,

$$\cdots \xrightarrow{S} \mathbb{Z}[C_p] \xrightarrow{\gamma-1} \mathbb{Z}[C_p] \xrightarrow{S} \mathbb{Z}[C_p] \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

with $S = \sum_{i=0}^{p-1} \gamma^i$. Applying $\operatorname{Hom}_{\mathbb{Z}[C_p]}(-,\mathbb{F}_p)$ (or tensoring with $\mathbb{F}_p$ and using trivial action) kills both $\gamma - 1$ and S, so all differentials vanish. Thus each cohomology group in positive degrees is 1-dimensional over $\mathbb{F}_p$.

The ring structures are given below.

$$H^*_{\text{grp}}(C_p;\mathbb{F}_p) \cong \begin{cases} \mathbb{F}_2[u], & |u| = 1, & \text{if } p = 2, \\ E(u)\otimes\mathbb{F}_p[t], & |u| = 1,\ |t| = 2,\ \beta(u) = t, & \text{if } p \neq 2, \end{cases}$$

$$\widehat{H}^*_{\text{grp}}(C_p;\mathbb{F}_p) \cong \begin{cases} \mathbb{F}_2[u, u^{-1}], & |u| = 1, & \text{if } p = 2, \\ E(u)\otimes\mathbb{F}_p[t, t^{-1}] & |u| = 1,\ |t| = 2,\ \beta(u) = t, & \text{if } p \neq 2. \end{cases}$$

So far, Tate cohomology is only defined for finite G. For compact Lie groups, we use the Tate construction:

Definition 3.3.11. Let G be a compact Lie group and M a G-module. The *Tate cohomology* of G with coefficients in M is $\widehat{H}^n_{\mathrm{grp}}(G;M) \coloneqq \pi_{-n} HM^{tG}$.

For finite G and naive G-spectra this definition agrees with the classical Tate construction [18].

Example 3.3.12. We compute $\widehat{H}^*_{\mathrm{grp}}(S^1;-)$ from $H^*_{\mathrm{grp}}(S^1;-) \cong H^*(\mathbb{C}P^\infty;-)$ with the periodicity class in degree 2 inverted. The Tate construction carries a periodicity class $u \in \pi_{-2}(Hk^{tS^1})$, and inverting u on $H^*_{\mathrm{grp}}(G;k)$ yields $\widehat{H}^*_{\mathrm{grp}}(G;k)$. So, over a field k, $\widehat{H}^*_{\mathrm{grp}}(S^1;k) \cong k[u,u^{-1}]$ with $|u| = 2$.

We will use Tate cohomology and these specific examples in the spectral sequences for computing TC in Chapter 4.

3.4 Mackey Functors

In stable homotopy theory, abelian groups are the coefficients we work with. In equivariant stable homotopy theory, Mackey functors play the role of abelian groups. It turns out that for a G-spectrum, the homotopy groups of X in some sense define a Mackey functor. In particular, we will see that $\pi_0^G(\mathbb{S})$ is given by the *Burnside Mackey functor*, which acts as $\mathbb{Z}$ in the equivariant context. We give two definitions of Mackey functors as both are beneficial, and then reconcile the definitions in the discussion following.

Definition 3.4.1 ([39, Def. 1.1])**.** A *G-Mackey functor* $\underline{M}$ consists of

- an abelian group $\underline{M}(G/H)$ for all $H \subseteq G$,
- restriction, transfer and conjugation homomorphisms

$$\begin{aligned} R^H_K &\colon \underline{M}(G/H) \to \underline{M}(G/K), \\ tr^K_H &\colon \underline{M}(G/K) \to \underline{M}(G/H), \\ c_g &\colon \underline{M}(G/H) \to \underline{M}(G/gHg^{-1}), \end{aligned}$$

  for all $K \subset H \subset G$ and $g \in G$, such that:

  - $R^H_H = tr^H_H = \mathrm{id}$ for all $H \subset G$,
  - $c_h \colon \underline{M}(G/H) \to \underline{M}(G/H)$ is the identity for all $H \subset G$ and $h \in H$,

- Functoriality: $R_H^L = R_H^K R_K^L$ and $tr_H^L = tr_K^L tr_H^K$ for all $H \subset K \subset L \subset G$,
- $c_g c_h = c_{gh}$ for all $g, h \in G$,
- Conjugation compatibility: $c_g R_K^H = R_{gKg^{-1}}^{gHg^{-1}} c_g$ and $c_g tr_K^H = tr_{gKg^{-1}}^{gHg^{-1}} c_g$ for all $K \subset H \subset G$ and $g \in G$,
- Double coset formula:

$$R_L^H tr_K^H = \sum_{x \in L \backslash H/K} tr_{L \cap xKx^{-1}}^L c_x R_{x^{-1}Lx \cap K}^K$$

  holds for all $K, L \subset H \subset G$.

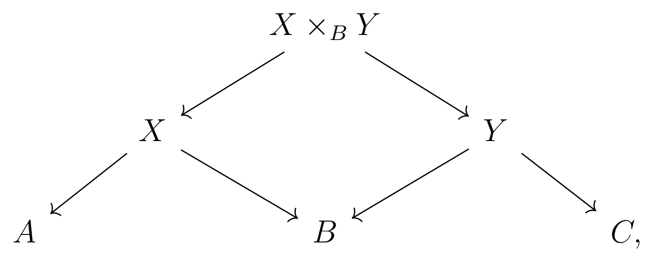
Note that the particular groups that we care about for this paper are $G = C_n$ and $G = S^1$, which are abelian groups. When restricting to abelian groups, the definition of Mackey functors simplifies a bit.

Remark 3.4.2. Let G be an abelian group, then every subgroup is fixed by conjugation, so the maps $c_g \colon \underline{M}(G/H) \to \underline{M}(G/H)$ are induced by multiplication by g on G/H. We also get that the double coset formula simplifies to

$$R_L^H tr_K^H = \sum_{x \in H/(LK)} tr_{L \cap K}^L R_{L \cap K}^K = [H \colon LK] \cdot (tr_{L \cap K}^L R_{L \cap K}^K).$$

Definition 3.4.1 is convenient for writing Mackey functors but inconvenient to check the axioms. When we write a Mackey functor, we do so using a diagram and the maps in Definition 3.4.1 as in Example 3.4.5. Checking that the maps satisfy the axioms is quite painful though, so we use a different definition of Mackey functors, which we can see to be equivalent. First, we define the domain for the functor.

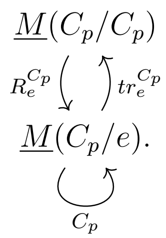
Definition 3.4.3. Span_G is the category whose objects are finite G-sets. A morphism $A \to B$ in Span_G is an isomorphism class of the diagram $A \leftarrow X \to B$ of G-equivariant maps of finite G-sets. Two spans $A \xleftarrow{r} X \xrightarrow{s} B$ and $A \xleftarrow{r'} X' \xrightarrow{s'} B$ are isomorphic if there is a G-equivariant bijection $\varphi \colon X \xrightarrow{\cong} X'$ as G-sets such that $r = r' \circ \varphi$ and $s = s' \circ \varphi$. Composition of $A \leftarrow X \to B$ and $B \leftarrow Y \to C$ is $A \leftarrow X \times_B Y \to C$ given by the pullback:



where $X \times_B Y$ is the usual set fiber product, with the diagonal G-action.

Definition 3.4.4. A *G-Mackey functor* $\underline{M}$ is a functor $\underline{M}\colon \mathrm{Span}_G \to \mathrm{Ab}$ such that $\underline{M}(A \sqcup B) = \underline{M}(A) \times \underline{M}(B)$.

Example 3.4.5. Let $G = C_p$, we draw a G-Mackey functor $\underline{M}$ as a diagram



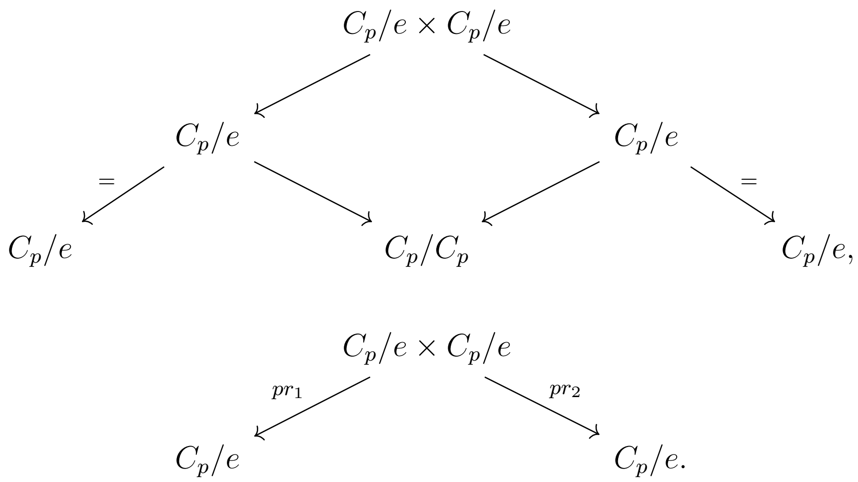
Often the C_p-action is omitted from diagrams, since the action is trivial in many cases that we care about.

We give an idea of how to reconcile the two definitions of Mackey functors below. First, note that we can write exactly the spans that correspond to each of the restriction, transfer and conjugation maps:

$$\begin{aligned} R^H_K &= \underline{M}(G/K \xleftarrow{\mathrm{id}} G/K \xrightarrow{pr} G/H), \\ tr^H_K &= \underline{M}(G/H \xleftarrow{pr} G/K \xrightarrow{\mathrm{id}} G/K), \\ c_g &= \underline{M}(G/H \xleftarrow{\mathrm{id}} G/gHg^{-1} \xrightarrow{\cdot g} G/gHg^{-1}). \end{aligned}$$

Then, the axioms come from the pullback squares that define composition in Span_G. We go through a sample computation of this for $G = C_p$.

Example 3.4.6. To compute $R^{C_p}_e tr^{C_p}_e(x)$ we look at the diagrams below:



There is an identification

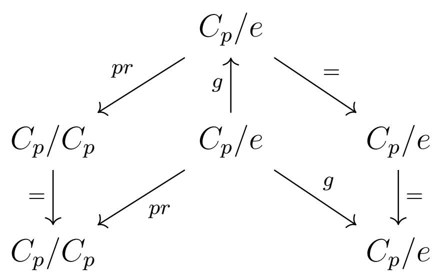
$$C_p/e \times C_p/e \xrightarrow{\cong} \coprod_{g \in C_p} C_p/e, \qquad (x,y) \to (x, x^{-1}y).$$

Here, the g-th summand is the orbit of (e,g). Under this identification, R_{pr_1} is the codiagonal $\coprod_{g \in C_p} \underline{M}(C_p/e) \to \underline{M}(C_p/e)$ where each summand maps by the identity on C_p/e, while tr_{pr_2} on the g-th summand is c_g followed by $tr_e^{C_p}$. Hence,

$$R_e^{C_p} tr_e^{C_p} = \sum_{g \in C_p} c_g,$$

which reduces to multiplication by p.

To see C_p acts trivially on restriction, we look at the diagram below.

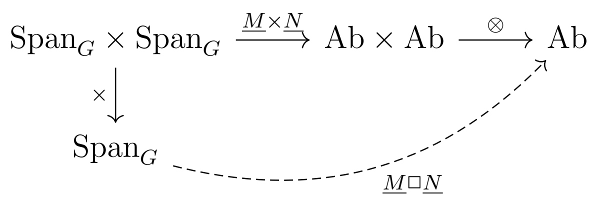


Isomorphic spans define the same morphism, so $c_g \circ R_e^{C_p} = R_e^{C_p}$.

Note here, we have concluded that c_g acts trivially on the image of $R_e^{C_p}$, but in general, c_g may act nontrivially on all of $\underline{M}(C_p/e)$.

Another benefit of the Span_G definition of Mackey functors, is that we can define a product on Mackey functors.

Definition 3.4.7. Span_G inherits a symmetric monoidal structure from finite G-sets. The *box product* $\underline{M} \square \underline{N}$ of G-Mackey functors is given by the left Kan extension of the below diagram.



We can also give a more concrete formula for the box product. For any finite G-set X, from [41, Lem. 2.20] we have the box product given by the coend

$$(\underline{M} \square \underline{N})(X) \cong \int^{(Y,Z) \in \mathrm{Span}_G \times \mathrm{Span}_G} \underline{M}(Y) \otimes \underline{N}(Z) \otimes \mathrm{Span}_G(X, Y \times Z).$$

This exact formula is unimportant to us, we can just think of this as the colimit

$$\underline{M}\square\underline{N}(X) = \varinjlim_{A,B\in \mathrm{Span}_G} \underline{M}(A)\otimes \underline{N}(B)$$

over all $f\colon A\times B\to X$ in Span_G.

Theorem 3.4.8 ([41, Thm. 2.24/Lem. 2.20])**.** *A map $f\colon \underline{M}\square\underline{N}\to\underline{P}$ of G-Mackey functors is equivalent to a* Dress pairing $\{f_H\colon \underline{M}(G/H)\otimes\underline{N}(G/H)\to \underline{P}(G/H)\}$ *satisfying so-called 'Frobenius relations'.*

The full description of these relations can be seen in [41, Thm. 2.24/Lem. 2.20]. The exact formulas and relations for computing box products and maps between them are not needed for our purposes.

We now go through some actual examples of Mackey functors.

Example 3.4.9. Let $G = C_p$. The *constant G-Mackey functor $\underline{\mathbb{Z}}$* is given by the diagram

$$\begin{array}{c}\mathbb{Z}\\ R_e^{C_p}=\mathrm{id}\downarrow\ \uparrow tr_e^{C_p}=\cdot p\\ \mathbb{Z}\end{array}$$

Since the restriction is surjective and $\mathbb{Z}$ is abelian, $c_g\colon\mathbb{Z}\to\mathbb{Z}$ acts by 1, so the C_p-action is trivial on the bottom level. Furthermore, we only need to check the easier set of axioms from Remark 3.4.2. Finally, since the only subgroups of C_p are e and C_p, functoriality and the double coset formula are immediate.

This example can be extended more generally. Let G be a finite group and let A be an abelian group. The constant Mackey functor $\underline{A}$ assigns A to each G/H_i with all restriction maps being the identity, and transfer maps $tr_{H_i}^{H_j}\colon G/H_i\to G/H_j$ given by multiplication by the index $[H_i : H_j]$.

Example 3.4.10. Let $G = C_p$. The constant opposite G-Mackey functor $\underline{\mathbb{Z}}$ is given by the diagram

$$\begin{array}{c}\mathbb{Z}\\ R_e^{C_p}=\cdot p\downarrow\ \uparrow tr_e^{C_p}=\mathrm{id}\\ \mathbb{Z}\end{array}$$

That is, just flipping the maps from Example 3.4.9.

Example 3.4.11 ([39, Def. 2.1.]). Let X be a G-spectrum. We get a G-Mackey functor $\underline{\pi}_n(X)$, with H-component given by

$$\underline{\pi}_n(X)(G/H) = [S^n \wedge G/H_+, X]^G = \pi_n^H(X).$$

The restrictions, transfers, and conjugations are defined by

$$\begin{aligned} R_K^H &= (\mathrm{id} \wedge pr_+)^*\colon [S^n \wedge G/H_+, X]^G \to [S^n \wedge G/K_+, X]^G, \\ tr_K^H &= (\mathrm{id} \wedge pr_+)_*\colon [S^n, X \wedge G/K_+]^G \to [S^n, X \wedge G/H_+]^G, \\ c_g &= (\mathrm{id} \wedge (-)g_+)^*\colon [S^n \wedge G/H_+, X]^G \to [S^n \wedge G/gHg_+^{-1}, X]^G. \end{aligned}$$

Here we use is an isomorphism $[S^n, X \wedge G/H_+]^G \cong [S^n \wedge G/H_+, X]^G$ by duality. We can see this from the equivalence $F_G(\Sigma^\infty G/H_+, X) \simeq \Sigma^\infty G/H_+ \wedge X$. Note that this is only a stable isomorphism, which is why we can not define a transfer map for X just a G-space.

Also note that for $G = e$, any G-Mackey functor $\underline{M}$ is just an abelian group M. By doing this, we can recover nonequivariant homotopy theory.

Since Mackey functors play the role of abelian groups in the equivariant stable setting, we are interested in what the unit would look like. In the nonequivariant setting, we have $\pi_0(\mathbb{S}) = \mathbb{Z}$, so the equivariant unit should be equivalent to $\underline{\pi}_0(\mathbb{S}_G)$.

Example 3.4.12. Write $\underline{\mathscr{A}}_G$ for the *Burnside Mackey functor for* G. $\underline{\mathscr{A}}_G$ assigns to G/H the Grothendieck group of finite H-sets under $\sqcup$. Then tr is given by induction of the group action and R is given by restriction of the group action. The box product makes the category of G-Mackey functors, Mack_G, into a closed symmetric monoidal category with the Burnside Mackey functor $\underline{\mathscr{A}}_G$ as the unit. We have $\underline{\pi}_0(\mathbb{S}_G) \cong \underline{\mathscr{A}}_G$ [25, Ch V, Prop. 9.10].

We give a concrete example of the Burnside Mackey functor for $G = C_p$.

Example 3.4.13. The Burnside Mackey functor for $G = C_p$ is

$$\underline{\mathscr{A}}_{C_p} = \begin{array}{c} \mathbb{Z}[C_p/C_p] \oplus \mathbb{Z}[C_p/e] \\ {\scriptstyle (1 \quad p)} \Big\downarrow \Big\uparrow {\scriptstyle \binom{0}{1}} \\ \mathbb{Z}. \end{array}$$

Since there are two conjugacy classes of subgroups of C_p, we get the top level. In fact, the top level has ring structure given by $\underline{\mathscr{A}}_{C_p}(C_p/C_p) \cong \mathbb{Z}[\alpha]/(\alpha^2 - p\alpha)$.

We will see in §4.5 that the Burnside Mackey functor for $G = C_{p^n}$ is related to the Witt vectors.

As we did in the nonequivariant case, we want to have equivariant Eilenberg-MacLane spaces and spectra as they are the building blocks of many constructions.

Definition 3.4.14. Let $\underline{M}$ be a G-Mackey functor. Define *Eilenberg-MacLane G-spaces*, denoted by $K(\underline{M}, n)$, to be such that

$$\underline{\pi}_i(K(\underline{M}, n)) = \begin{cases} \underline{M} & i = n \\ 0 & \text{else.} \end{cases}$$

Define an *equivariant Eilenberg-MacLane spectrum $H\underline{M}$* such that

$$\underline{\pi}_n(H\underline{M}) = \begin{cases} \underline{M} & n = 0 \\ 0 & \text{else,} \end{cases}$$

where $H\underline{M}_n \simeq K(\underline{M}, n)$.

It is important to note that this definition used grading on n and *not* on V. It turns out that $\underline{\pi}_V(H\underline{M})$ says something about the homology and cohomology of representation spheres which are complicated, and in particular nonzero. For further discussion on this, and computations, see [42].

Now that we have equivariant Eilenberg-MacLane spectra, we could instead define the box product of Mackey functors as $\underline{M} \square \underline{N} = \underline{\pi}_0(H\underline{M} \wedge H\underline{N})$ [24, Lem. 4].

Recall from §2.2 that Eilenberg-MacLane spectra define (co)homology theories. Now with the equivariant analogues, we can move these notions to the equivariant context with the below definition.

Definition 3.4.15. Given a G-Mackey functor $\underline{M}$, define *equivariant homology* with coefficients in $\underline{M}$ as

$$H_k^G(X; \underline{M}) \coloneqq \underline{\pi}_k^G(H\underline{M} \wedge X).$$

Define *equivariant cohomology* with coefficients in $\underline{M}$ as homotopy classes of G-equivariant maps

$$H^k_G(X; \underline{M}) \coloneqq [X, \Sigma^k H\underline{M}]^G.$$

As a special case, we can recover the nonequivariant versions.

Example 3.4.16. Following on from Example 2.2.24, setting $G = e$ recovers the underlying abelian group M and the nonequivariant homology and cohomology.

3.5 Green and Tambara Functors

Nonequivariantly, ring spectra provide multiplicative structure. Equivariantly, we want the analogous multiplicative structure on Mackey functors. The box product on Mack_G gives a symmetric monoidal structure, and *Green functors* are the commutative monoids for this product.

Definition 3.5.1. Let G be a finite group. A *G–Green functor* $\underline{M}$ is a commutative monoid with respect to the box product $\square$ in Mack_G. Write Green_G for the category of G–Green functors. Equivalently, $\underline{M}$ is a G–Mackey functor equipped with a commutative ring structure at each level $\underline{M}(G/H)$, natural in finite G–sets G/H.

As with Mackey functors, it is useful to have an explicit axiomatic presentation.

Definition 3.5.2. A *G–Green functor* $\underline{M}$ consists of a G–Mackey functor together with, for every subgroup $H \subset G$, an associative, commutative, unital ring structure on $\underline{M}(G/H)$ such that $R^H_K\colon \underline{M}(G/H) \to \underline{M}(G/K)$ and Frobenius reciprocity: For $K \subset H$, the transfer $tr^H_K : \underline{M}(G/K) \to \underline{M}(G/H)$ is $\underline{M}(G/H)$–linear when $\underline{M}(G/K)$ is viewed as a left $\underline{M}(G/H)$–module via R^H_K:

$$tr^H_K\big(a \cdot R^H_K(b)\big) = tr^H_K(a) \cdot b, \qquad \text{for } a \in \underline{M}(G/K),\ b \in \underline{M}(G/H).$$

Example 3.5.3. Following on from Example 3.4.9, the constant C_p-Mackey functor $\underline{\mathbb{Z}}$ is a C_p–Green functor because $\mathbb{Z}$ is a commutative ring. Since $R = \mathrm{id}$ and $tr^{C_p}_e$ is multiplication by p, Frobenius reciprocity is immediate:

$$tr^{C_p}_e\left(a \cdot R^{C_p}_e(b)\right) = pab = tr^{C_p}_e(a) \cdot b.$$

Green functors are the *additively* commutative ring objects in Mack_G. *Tambara functors* refine this by also encoding *multiplicative transfers*, called norms. Intuitively, a Tambara functor sees not just the additive transfer along a finite G–map, but also the multiplicative one, and the two interact via distributivity.

Remark 3.5.4. If A is a $\mathbb{Z}[G]$–module, then $G/H \mapsto A^H$ defines a Mackey functor with restriction given by the inclusion $A^H \hookrightarrow A^K$ for $K \subset H$ and transfer $tr^H_K(x) = \sum_{[h]\in H/K} h \cdot x$. Moreover, if A is a *commutative* $\mathbb{Z}[G]$–algebra, one also has *norms*

$$N^H_K(x) = \prod_{[h]\in H/K} h \cdot x$$

which will satisfy the Tambara axioms.

As with Mackey functors, we have both an axiomatic and a categorical description.

Definition 3.5.5 ([43, Prop. 6.9])**.** A *G–Tambara functor* $\underline{R}$ is a G–Green functor together with *norm maps* for each $K \subset H \subset G$,

$$N_K^H : \underline{R}(G/K) \longrightarrow \underline{R}(G/H)$$

satisfying the additional axioms:

- Functoriality: $N_H^H = \mathrm{id}$ and $N_K^H \circ N_J^K = N_J^H$ for $J \subset K \subset H$,
- Multiplicativity: $N_K^H(ab) = N_K^H(a)N_K^H(b)$,
- Power/Restriction relation: For $K \subset H$,
$$R_K^H\big(N_K^H(a)\big) = \prod_{[h]\in H/K} c_h(a).$$
More generally, for $J \subset H$ one has the double–coset (Beck–Chevalley) formula.
- Frobenius reciprocity: For $K \subset H \subset L$ and $a \in \underline{R}(G/H)$, $b \in \underline{R}(G/K)$,
$$N_K^L\left(R_K^H(a) \cdot b\right) = a^{|L:H|} \cdot N_K^L(b),$$
- Tambara distributivity: Norms distribute over sums via a universal polynomial determined by the H–set H/K.

For Tambara distributivity, we avoid writing the polynomial explicitly as it is complicated in general, and outside the scope of this thesis. For reference, see [43, Prop. 6.9]

Definition 3.5.6. Let G be a finite group. The category Bispan_G has finite G–sets as objects and a morphism $X \to Y$ in Bispan_G is an isomorphism class of a diagram of G–maps $X \xleftarrow{s} A \xrightarrow{m} B \xrightarrow{t} Y$ called a *bispan.* Composition is defined up to isomorphism by first forming the pullback, then using the "exponential (distributor) diagram" so that disjoint union is the coproduct and cartesian product is the product (see [15] or [45, §4.1] for the explicit construction).

With this categorical setup, we give another definition of Tambara functors.

Definition 3.5.7. A *G–Tambara functor* is a functor $\underline{R}\colon \mathrm{Bispan}_G \to \mathrm{Set}$ such that $\underline{R}(A \sqcup B) \cong \underline{R}(A) \times \underline{R}(B)$ and $\underline{R}(\emptyset) = \{*\}$.

A Tambara functor is essentially a ring object in Mackey functors (a "Green functor with norms"). Note that restricting to bispans of the form $X \leftarrow A \rightarrow A \rightarrow Y$ recovers the underlying Mackey functor structure.

From this categorical viewpoint, R comes from pullback, tr from additive pushforward, N from multiplicative pushforward, and Tambara's distributivity is precisely functoriality for bispan composition.

Example 3.5.8. Following on from Example 3.5.3, let A be a commutative ring. The constant C_p-Tambara functor $\underline{A}$ has value A at both orbits, with the extra norm map $N(x) = x^p$. In a diagram this is written

$$\begin{array}{c} A \\ {\scriptstyle N=()^p}\curvearrowright \big\downarrow {\scriptstyle \mathrm{id}} \curvearrowleft {\scriptstyle tr=\cdot p} \\ A. \end{array}$$

Not every Mackey functor can be given the structure of a Tambara functor.

Example 3.5.9 (Non-example)**.** Following on from Example 3.4.10, let $G = C_p$ and A a commutative ring. If we try to make the *opposite* constant Mackey functor into a Tambara functor by setting $R = \cdot p$, $tr = \mathrm{id}$, and $N = (-)^p$, the power relation forces $RN(1) = 1^p = 1$, so $pN(1) = 1$. This means that $N(1) = \frac{1}{p}$, which is impossible unless p is invertible in A. For example, $A = \mathbb{Z}$ fails. Hence no Tambara structure exists in general.

Example 3.5.10. Following on from Example 3.4.13, the Burnside Mackey functor $\underline{\mathscr{A}}$ is a Tambara functor. For $H \subset G$ and a finite H–set X, the norm is

$$N_H^G(X) = \prod_{[g] \in G/H} gX$$

with G permuting the factors by left translation. For the general case of the Tambara norm formula, see [43, §2]. We derive the formula for $G = C_p$ and $H = e$. Suppose $|X| = n$, then the G–set $\mathrm{Hom}(C_p, X)$ decomposes into n fixed points (constant functions) and $\frac{n^p - n}{p}$ free p–orbits. Thus in the Burnside ring

$$N_e^{C_p}(X) = n[C_p/C_p] + \frac{n^p - n}{p}[C_p/e] \in \underline{\mathscr{A}}(C_p/C_p).$$

It turns out that we can get Tambara functors by taking $\underline{\pi}_0$ of G-spectra. Not only this, but also that *every* Tambara functor arises in this way.

Theorem 3.5.11 ([44, Thm. 1.1])**.** *Every Tambara functor occurs as* $\underline{\pi}_0$ *of a commutative ring spectrum.*

This shows that equivariant stable homotopy theory and Tambara functors are deeply interwoven areas of mathematics.

Tambara functors abstract the restriction, transfer, and norm patterns that occur on $\underline{\pi}_0$ of genuine G–ring spectra. To build these multiplicative transfers at the spectra level we now construct the norm functor $N_H^G : \mathrm{Sp}^H \to \mathrm{Sp}^G$, whose $\underline{\pi}_0$ recovers the Tambara norms.

3.6 The Norm Functor

The norm functor plays a central role in modern equivariant stable homotopy theory. In particular, it is essential to the work of Hill, Hopkins, and Ravenel on the Kervaire invariant one problem [22], where norms provide the extra multiplicative structure required to construct equivariant analogues of classical spectra. Thus, understanding norms is not only of technical interest but also of deep conceptual importance in how multiplicative structure interacts with equivariance. In this section, we discuss the high level of the construction of the norm functor $N_H^G \colon \mathrm{Sp}^H \to \mathrm{Sp}^G$.

We start with the induction map on representations

$$\begin{aligned} \mathrm{ind}_H^G \colon H\text{-Rep} &\to G\text{-Rep} \\ V &\mapsto \bigoplus_{[G:H]} V. \end{aligned}$$

Since this functor is meant to give a G-representation, we need to define the G-action. Pick coset representatives $\{g_1 = e, g_2, ..., g_n\}$ for G/H. We then have a G-action given by $gg_i = g_{\sigma(i)}h_i$ for some permutation σ. Let V be an H-representation, then the g action on V is given by

$$g(v_1, ..., v_n) = (h_{\sigma^{-1}(1)}v_{\sigma^{-1}(1)}, ..., h_{\sigma^{-1}(n)}v_{\sigma^{-1}(n)}).$$

In particular, $g(0, ..., v, ..., 0) = (0, ..., h_i v, ..., 0)$ with v in i-th position and $h_i v$ in $\sigma(i)$-th position. Note that the labeling from coset choices gives canonically isomorphic G-representations (and soon G-spectra). The construction is natural in the orbit G/H, hence independent up to canonical isomorphism.

The norm on representations is defined similarly but with tensor products

$$\begin{aligned} N_H^G \colon H\text{-Rep} &\to G\text{-Rep} \\ V &\mapsto \bigotimes_{[G:H]} V \end{aligned}$$

with G acting by permuting tensor factors using the same coset decomposition.

Passing to spectra, the *spectral norm* is defined by

$$\begin{aligned} N_H^G \colon \mathrm{Sp}^H &\to \mathrm{Sp}^G \\ X &\mapsto \bigwedge_{[G:H]} X \end{aligned}$$

again with the G-action given by permuting smash factors according to the same coset decomposition.

Theorem 3.6.1 ([22, Prop. 9.7.4/Cor. 10.7.4])**.** *The norm functor $N_H^G \colon \mathrm{Sp}^H \to \mathrm{Sp}^G$ is symmetric monoidal and sends commutative ring spectra to commutative ring spectra.*

There are two other canonical functors going between G-spectra and H-spectra: the transfer and the forgetful functor. The transfer is essentially an additive version of the norm, given by

$$\begin{aligned} tr_H^G \colon Sp^H &\to Sp^G \\ X &\mapsto \bigvee_{[G:H]} X. \end{aligned}$$

The forgetful functor is $i_H^G \colon \mathrm{Sp}^G \to \mathrm{Sp}^H$ where $i_H^G X \simeq X$ but only remembering the H-action on X.

Theorem 3.6.2. *When restricted to commutative ring spectra, there is an adjunction*

$$N_H^G \colon \mathrm{CommRingSp}^H \rightleftarrows \mathrm{CommRingSp}^G \colon i_H^G.$$

Proof. The idea for this proof is that N_H^G is a multiplicative analogue of coinduction. We will omit the H and G scripts on i and N to clean up notation. Given $A \in \mathrm{CommRingSp}^H$, the unit map $\iota_A \colon A \to iNA$ is the smash factor indexed by the identity coset $eH \subset G/H$. Since A sits as a factor in a smash of commutative ring spectra, ι_A is a map of commutative H-ring spectra. Define

$$\mu \colon NiB = \bigwedge_{[G:H]} B \xrightarrow{\text{iterated } \mu_B} B$$

given by smashing together $[G : H]$ copies of the ring multiplication $\mu_B \colon B \wedge B \to B$. Since B is commutative, μ is a G-map.

Take $A \in \mathrm{CommRingSp}^H$, $B \in \mathrm{CommRingSp}^G$. For a map $f \colon NA \to B$ in $\mathrm{CommRingSp}^G$, define $\Phi(f) \coloneqq if \circ \iota_A \colon A \to iB$. Both components are ring maps, so $\Phi(f)$ lies in $\mathrm{CommRingSp}^H(A, iB)$.

For $g\colon A \to iB$ in $\mathrm{CommRingSp}^H$, we get $Ng\colon NA \to NiB$, and define $\Psi(g) \coloneqq \mu \circ Ng\colon NA \to B$. Consider the composition

$$NA \xrightarrow{N\iota_A} NiNA \xrightarrow{N(if)} NiB \xrightarrow{\mu} B.$$

Inside the $NiNA$, the identity coset factor is sent by if to the corresponding factor in NiB. All other factors land in B but are multiplied by μ. Associativity and unitality of m collapse the composite back to f. On the other hand, we have

$$A \xrightarrow{\iota_A} iNA \xrightarrow{i(\mu \circ Ng)} iB.$$

The restriction $i\mu$ only sees the identity coset smash factor, so the composite is exactly g. Thus, Φ and Ψ are mutually inverse natural bijections. This proves the adjunction.

From all this, we also get that $\eta\colon A \xrightarrow{\iota_A} iNA$ is the unit and $\varepsilon\colon NiB \xrightarrow{\mu} B$ is the counit. In this argument, naturality in A and B follows from functoriality of ι and μ. □

This adjunction requires commutative ring spectra. We can see Theorem 3.6.2 does not apply for noncommutative ring spectra.

Example 3.6.3. Let $G = C_2 = \langle \gamma \mid \gamma^2 = e \rangle$, $H = e$, and A a noncommutative ring spectrum with multiplication map $\mu\colon A \wedge A \to A$. Then $N_e^{C_2} A = A \wedge A$ with C_2 acting by swapping the smash factors. From Theorem 3.6.2, we would have $\mathrm{Hom}_{\mathrm{AssRing}}(A, i_e^{C_2} B) \cong \mathrm{Hom}_{\mathrm{AssRing}^{C_2}}(N_e^{C_2} A, B)$. Given $\varphi\colon A \to i_e^{C_2} B$ a ring map, we should get a C_2-map $N_e^{C_2} A \to B$ given by $A \wedge A \xrightarrow{\mu} A \xrightarrow{\varphi} B$. Since μ is not commutative, $\mu\tau \neq \mu$ where τ is the swap map. Note that in this case, the C_2-action is just given by this swap map. For μ to be equivariant, we need $\mu\gamma = \gamma\mu = \mu$. Thus, μ is not equivariant so we have a contradiction.

With this spectral norm, we can define a norm on Mackey functors.

Definition 3.6.4. Let $H \subset G$, the *Mackey functor norm functor* $N_H^G\colon \mathrm{Mack}_H \to \mathrm{Mack}_G$ is given by $N_H^G \underline{M} \coloneqq \underline{\pi}_0(N_H^G H\underline{M})$.

We return to spectral and Mackey functor norms in Chapter 5.

Chapter 4

Topological Hochschild Homology

Topological Hochschild homology (THH) plays a central role in modern algebraic topology and algebraic K-theory. It can be approached from several perspectives: as a refinement of Hochschild homology for ring spectra, as the target of trace maps, or through its description as the geometric realisation of a cyclic bar construction. In this chapter, we develop the simplicial and cyclic language to define THH, before turning to its more structural properties.

4.1 Simplicial Background

We recall and introduce the simplex and cyclic categories where a lot of our objects will be functors out of. Throughout this section, we mainly focus on simplicial sets, but these can be generalised to simplicial objects in a suitable category. In particular, analogous statements can all be made for simplicial spectra. If there are any technical differences, we give the definitions alongside each other.

Definition 4.1.1. Write Δ for the simplex category with objects $[n] = \{0 < 1 < \cdots < n\}$ and morphisms given by weakly order-preserving maps. A *simplicial set* is a functor $X_\bullet \colon \Delta^{\mathrm{op}} \to \mathrm{Set}$. That is, X is a sequence of sets $\{X_n\}_{n\geq 0}$ with *face maps* $d_i \colon X_n \to X_{n-1}$ and *degeneracy maps* $s_i \colon X_n \to X_{n+1}$ with $0 \leq i \leq n$, satisfying the relations:

$$d_i d_j = d_{j-1} d_i, \text{ if } i < j, \quad s_i s_j = s_{j+1} s_i, \text{ if } i \leq j, \quad d_i s_j = \begin{cases} s_{j-1} d_i & \text{if } i < j, \\ s_j d_{i-1} & \text{if } i > j+1, \\ \mathrm{id} & \text{else.} \end{cases}$$

Example 4.1.2. Let $S^1_\bullet$ be the simplicial set defined by $S^1_n := \mathbb{Z}/(n+1)$, for $n \geq 0$. We write an element of S^1_n as k, taken modulo $n+1$. The face and degeneracy maps are

$$d_i(k) = \begin{cases} k & \text{for } k < i, \\ k-1 & \text{for } k \geq i, \end{cases} \qquad s_i(k) = \begin{cases} k & \text{for } k \leq i, \\ k+1 & \text{for } k > i, \end{cases}$$

where arithmetic is taken modulo $n+1$. These formulas remove or repeat the i-th vertex of an $(n+1)$-gon. Equivalently, there is an isomorphism of simplicial sets $S^1_\bullet \cong \Delta^1_\bullet/\partial\Delta^1_\bullet$ obtained by identifying the two endpoints of Δ^1 (so $S^1_\bullet$ has exactly one nondegenerate 0–simplex and one nondegenerate 1–simplex and all higher simplices are degeneracies).

Definition 4.1.3. The *(Connes) cyclic category* Λ is an extension of the simplicial category Δ. It has the same objects but the morphisms are extended by the cyclic permutation $\tau_n \colon [n] \to [n]$ with the extra relations

$$\tau_n^{n+1} = \mathrm{id}, \quad d_i\tau_n = \begin{cases} \tau_{n-1}d_{i-1} & 1 \leq i \leq n \\ d_n & i = 0 \end{cases}, \quad s_i\tau_n = \begin{cases} \tau_{n+1}s_{i-1} & 1 \leq i \leq n \\ \tau^2_{n+1}s_n & i = 0. \end{cases}$$

The *r-cyclic category* Λ_r is an extension of Δ with the morphisms $\tau_n \colon [n] \to [n]$ subject to the relations for Λ, except $\tau_n^{r(n+1)} = \mathrm{id}$ instead of $\tau_n^{n+1} = \mathrm{id}$.

Definition 4.1.4. An *r-cyclic set* is a functor from the cyclic category $X_\bullet \colon \Lambda_r^{\mathrm{op}} \to$ Set. That is, X is a simplicial set together with maps $\tau_n \colon X_n \to X_n$ satisfying the previous relations. In particular, a *cyclic set* is when $r = 1$, and has a cyclic C_{n+1}-action on each X_n.

Example 4.1.5. Following on from Example 4.1.2, each $S^1_n = \mathbb{Z}/(n+1)$ carries a free and transitive action of the cyclic group C_{n+1} by addition. So $S^1_\bullet$ becomes a cyclic set.

Remark 4.1.6. For a category $\mathcal{C}$, more generally, we have simplicial and r-cyclic objects $X_\bullet \colon \Delta^{\mathrm{op}} \to \mathcal{C}$ and $X_\bullet \colon \Lambda_r^{\mathrm{op}} \to \mathcal{C}$. These satisfy the same relations that we demanded of simplicial and r-cyclic sets. In particular, we will extend to spectra.

We give a way of turning a simplicial set into a topological space.

Definition 4.1.7. The *geometric realisation* $|{-}| \colon \mathrm{sSet} \to \mathrm{Top}$ of a simplicial set $X_\bullet$ is

$$|X| = \left(\coprod_{n \geq 0} X_n \times \Delta^n \right) \Big/ \sim$$

where the relation is gluing along the face and degeneracy maps $(x, d_i t) \sim (d_i x, t), (x, s_i t) \sim (s_i x, t)$.

Similarly, the geometric realisation $|-|\colon \mathrm{sSp} \to \mathrm{Sp}$ of a simplicial spectrum $X_\bullet$ is

$$|X| = \left(\coprod_{n \geq 0} X_n \wedge \Delta^n_+ \right) \Big/ \sim$$

with the same relations.

Example 4.1.8. Following on from Example 4.1.5, by applying the geometric realisation to $S^1_\bullet$ we obtain

$$|S^1_\bullet| = \left(\coprod_{n \geq 0} (\mathbb{Z}/(n+1)) \times \Delta^n \right) \Big/ \sim .$$

Gluing along the face maps identifies the two endpoints of every 1–simplex, while the degeneracies identify all 0–simplices, while higher simplices are degenerate and glue along their faces. Thus $|S^1_\bullet|$ is a CW complex with one 0–cell and one 1–cell attached by the map sending both endpoints to the unique 0–cell, that is, a circle.

Equivalently, since $|-|$ preserves colimits, [38]

$$|S^1_\bullet| \cong \left|\Delta^1_\bullet / \partial \Delta^1_\bullet\right| \cong |\Delta^1_\bullet| / |\partial \Delta^1_\bullet| \cong \Delta^1 / \partial \Delta^1 \cong S^1.$$

Hence the geometric realisation of the simplicial circle is homeomorphic to the usual topological circle.

When we take the geometric realisation of not just a simplicial set, but a cyclic set, we get extra structure on the resulting space.

Theorem 4.1.9. *[9, Lem 1.6] Let $X_\bullet \colon \Lambda^{\mathrm{op}} \to \mathrm{Set}$ be a cyclic set. The geometric realisation $|X_\bullet|$ has an S^1-action.*

Proof. For $n \geq 0$, write $[n] = \{0, ..., n\}$. Consider $\mathcal{C}_\bullet \colon \Delta^{\mathrm{op}} \to \mathrm{Top}$ by

$$[n] \mapsto \mathcal{C}_n = \left\{ (\theta; t_0, ..., t_n) \in S^1 \times I^{n+1} \,\middle|\, \sum_{i=0}^{n} t_i = 1 \right\}.$$

We think of $(\theta; t_0, ..., t_n)$ as a necklace on S^1, start at a basepoint $\theta \in S^1$ and go anticlockwise, laying down consecutive arcs of length t_i that partition the circle. We define the face and degeneracy maps by merging or inserting consecutive arcs and the cyclic operator by rotating the labels and shifting the basepoint. We use $S^1 \cong \mathbb{R}/\mathbb{Z}$,

$$d_i(\theta; t_0, ..., t_n) = \begin{cases} (\theta; t_0, ..., t_i + t_{i+1}, ..., t_n) & 0 \leq i < n, \\ (\theta; t_n + t_0, t_1, ..., t_{n-1}) & i = n, \end{cases}$$
$$s_i(\theta; t_0, ..., t_n) = (\theta; t_0, ..., t_i, 0, t_{i+1}, ..., t_n),$$
$$\tau_n(\theta; t_0, ..., t_n) = (\theta + t_n; t_n, t_0, ..., t_{n-1}).$$

Then, it is simple to check the compatibility of these maps.

There is a natural identification $|\Lambda(-, [n])| \cong \mathcal{C}_n$ natural in $[n]$. That is, $\mathcal{C}_\bullet$ is a geometric realisation of the so-called *universal cyclic simplex* $\Lambda(-, [\bullet])$. Note that S^1 acts on $\mathcal{C}_\bullet$ by rotating the basepoint only: for $\alpha \in S^1$,

$$\alpha \cdot (\theta; , t_0, ..., t_n) = (\theta + \alpha; t_0, ..., t_n).$$

We claim that this action commutes with all other maps d_i, s_i, τ_n. Those maps never change the multiset of arc lengths and rotating by α is commutative since addition in S^1 is commutative. Therefore, we get a canonical S^1-action on $|\Lambda(-, [\bullet])|$.

Let $X : \Lambda^{\mathrm{op}} \to \mathrm{Set}$ be a cyclic set. Then $|X| \cong \left(\coprod_{n \geq 0} X_n \times \mathcal{C}_n\right) / \sim$. Since the S^1-action on $\mathcal{C}_\bullet$ commutes with all structure maps, the formula $\alpha \cdot [x, c] = [x, \alpha \cdot c]$ is well-defined on the quotient, and gives a continuous S^1-action on $|X|$. □

In fact, only a small adjustment to the above proof gives us the following theorem.

Theorem 4.1.10 ([9, Lem. 1.6]). *Let $X_\bullet : \Lambda_r^{\mathrm{op}} \to \mathrm{Set}$ be an r-cyclic set, then $|X_\bullet|$ has an S^1-action.*

The proof of this follows the same outline as before to find an $\mathbb{R}/r\mathbb{Z}$-action, which gives an S^1-action once we identify $\theta + r\mathbb{Z}$ with $e^{2\pi i\theta/r}$. There are some small subtleties with how we should geometrically realise an r-cyclic set: in the standard way with Δ^n or with Λ_r^n. It turns out that they are equivalent [9, Lem. 1.8].

These theorems and proofs both hold in the case of cyclic spectra, which we will make use of later in this chapter.

It is convenient to split up a simplicial set into smaller pieces that glue together in the expected way.

Definition 4.1.11 ([9, §1]). Let $X_\bullet\colon \Delta^{\mathrm{op}} \to \mathrm{Set}$ be a simplicial set and let $r \in \mathbb{N}$. The *edgewise subdivision* of $X_\bullet$ is $sd_r X_\bullet$ with $sd_r X_n = X_{(n+1)r-1}$, and with face and degeneracy maps $d_i^{sd}\colon sd_r X_n \to sd_r X_{n-1}$ and $s_i^{sd}\colon sd_r X_n \to sd_r X_{n+1}$ given by:

$$d_i^{sd} = d_i \circ d_{i+(n+1)} \circ \ldots \circ d_{i+(r-1)(n+1)}, \qquad s_i^{sd} = s_{i+(r-1)(n+2)} \circ \ldots \circ s_{i+(n+2)} \circ s_i.$$

Example 4.1.12. Following on from Example 4.1.8, for the simplicial circle $S^1_\bullet$, the subdivision $sd_r S^1_\bullet$ corresponds to refining the polygon model of S^1 by a factor of r. Each n-simplex is subdivided into r smaller simplices, which geometrically encodes an r-fold covering of the circle.

We look at subdivision because it behaves nicely with geometric realisation, and can make some computations easier.

Theorem 4.1.13 ([9, Lem. 1.1]). *The map $D_r\colon |sd_r X_\bullet| \to |X_\bullet|$ of geometric realisations induced from the diagonal embedding $1 \times \mathrm{diag}_r\colon X_{rm-1} \times \Delta^{m-1} \to X_{rm-1} \times \Delta^{rm-1}$ is a homeomorphism.*

Similarly, subdivision can be performed on simplicial spectra, and the same theorem applies in this context.

4.2 Topological Hochschild Homology

Before passing to spectra, we recall the purely algebraic definition of Hochschild homology, an invariant of rings that captures how far the ring is from being commutative.

Definition 4.2.1. Let R be an associative ring. The *Hochschild complex* is a simplicial abelian group $C_\bullet(R)$, with $C_n(R) = R^{\otimes(n+1)}$. Let σ be the cyclic permutation moving the last tensor factor to the front. The face and degeneracy maps are as follows:

$$d_i = \begin{cases} \mathrm{id}^{\otimes i} \otimes \mu \otimes \mathrm{id}^{\otimes(n-i-1)} & \text{for } 0 \le i < n, \\ (\mu \otimes \mathrm{id}^{\otimes(n-1)}) \circ \sigma & \text{for } i = n, \end{cases} \qquad s_i = \mathrm{id}^{\otimes(i+1)} \otimes \eta \otimes \mathrm{id}^{\otimes(n-i)}.$$

The *Hochschild homology* of R, $\mathrm{HH}(R)$, is the homology of the Hochschild complex, $\mathrm{HH}_*(R) = H_*(C_\bullet, d_\bullet)$.

For our purposes, we want to consider the so-called *Shukla homology*, the derived version of Hochschild homology. For Shukla homology, let k be a field, we take R to be an associative k-algebra with $\otimes_k$ in place of $\otimes$. As is standard in the literature, we write Hochschild homology for Shukla homology.

We can extend the definition of HH to have more general coefficients.

Definition 4.2.2. Let R be an associative ring and let M be an (R, R)-bimodule. The *Hochschild complex with coefficients in* M is a simplicial complex $C_\bullet^M(R)$ with $C_n^M(R) = M \otimes R^{\otimes n}$. The *Hochschild homology with coefficients in* M is $\mathrm{HH}_*(R; M) \coloneqq H_*(C_\bullet^M(R), d_\bullet)$.

Lemma 4.2.3 ([33, Rmk. 2.4])**.** *Let* R *be an associative ring, then*

$$\mathrm{HH}_*(R) \cong \mathrm{Tor}_*^{R\otimes R^{\mathrm{op}}}(R, R).$$

Thus HH is related to homological algebra over the enveloping algebra $R \otimes R^{\mathrm{op}}$.

Example 4.2.4 ([23, Prop. 2.6])**.** $\mathrm{HH}_*(\mathbb{F}_p) \cong \mathbb{F}_p\langle\mu\rangle$, the free divided power algebra on μ with $|\mu| = 2$.

Since $\mathbb{F}_p$ is not flat over $\mathbb{Z}$, doing the computation of $\mathrm{HH}_*(F_p)$ in the nonderived sense would yield only $\mathbb{F}_p$.

We will use Example 4.2.4 as something to refer back to when we compute $\mathrm{THH}(\mathbb{F}_p)$ to see the difference.

4.2.1 From algebra to topology

To extend this theory, we replace tensor products by smash products, rings by ring spectra and modules by module spectra. This shift allows THH to interact with trace methods in algebraic K-theory and leads to computationally accessible invariants, which turn out to be simpler than their algebraic analogues, as we will see in the case of $\mathrm{THH}(F_p)$.

Definition 4.2.5. Let R be a ring spectrum with multiplication map μ and unit map η. The *cyclic bar construction* is the simplicial spectrum $B_\bullet^{\mathrm{cy}}(R)\colon \Delta^{\mathrm{op}} \to \mathrm{Sp}$ given by $[n] \mapsto R^{\wedge(n+1)}$. Let σ be the cyclic permutation moving the last smash factor to the front. The face and degeneracy maps are as follows:

$$d_i = \begin{cases} \mathrm{id}^{\wedge i} \wedge \mu \wedge \mathrm{id}^{\wedge(n-i-1)} & \text{for } 0 \le i < n, \\ (\mu \wedge \mathrm{id}^{\wedge(n-1)}) \circ \sigma & \text{for } i = n, \end{cases} \qquad s_i = \mathrm{id}^{\wedge(i+1)} \wedge \eta \wedge \mathrm{id}^{\wedge(n-i)}.$$

The *topological Hochschild homology* of R, $\mathrm{THH}(R)$, is the geometric realisation of the cyclic bar construction: $\mathrm{THH}(R) \coloneqq |B_\bullet^{\mathrm{cy}}(R)|$.

As a special case, if R is just a ring, we take $\mathrm{THH}(R) := \mathrm{THH}(HR)$. Thus, THH takes more general inputs than HH.

Lemma 4.2.6. *Let R be a ring spectrum, then $\mathrm{THH}(R)$ admits a canonical S^1-action.*

Proof. The cyclic bar construction is a cyclic object, with C_n-action at each level given by τ_n cyclically permuting the $(n+1)$-smash factors. Then by Theorem 4.1.9, $\mathrm{THH}(R) = |B^{\mathrm{cy}}_\bullet(R)|$ admits a canonical S^1-action. □

In the commutative case, this action is seen through the following theorem:

Theorem 4.2.7 ([31])**.** *If R is a commutative ring spectrum, then there is an equivalence $\mathrm{THH}(R) \simeq (R \otimes S^1_\bullet)$.*

Remark 4.2.8. More generally, for M and N left and right R-modules respectively, we can also define a two sided bar construction, $B_\bullet(M, R, N)\colon \Delta^{\mathrm{op}} \to \mathrm{Sp}$ given by $[n] \mapsto M \wedge R^{\wedge n} \wedge N$.

With this, we can generalise the definition of THH to have more general coefficients.

Definition 4.2.9. Let R be a ring spectrum and M be an (R, R)-bimodule. The *cyclic bar construction with coefficients in M* is the simplicial spectrum $B^{\mathrm{cy}}_\bullet(R; M)\colon \Delta^{\mathrm{op}} \to \mathrm{Sp}$ given by $[n] \mapsto M \wedge R^{\wedge n}$. The *topological Hochschild homology with coefficients in M* is $\mathrm{THH}(R; M) := |B^{\mathrm{cy}}_\bullet(R; M)|$.

Remark 4.2.10. In this case, $\mathrm{THH}(R; M)$ does *not* admit an S^1-action, since the cyclic bar construction with coefficients is not a cyclic object (despite the name).

We can think of $\mathrm{THH}(R; M)$ in another way. Write R^{op} for R with opposite multiplication. That is, ab in R turns into ba in R^{op}. Then an (R, R)-bimodule is equivalent to an $R \wedge R^{\mathrm{op}}$-module. Define the derived smash product $\wedge^L$ by replacing at least one smash factor with a cofibrant $R \wedge R^{\mathrm{op}}$-module. With these definitions, we have the following lemma:

Lemma 4.2.11. *Let R be a ring spectrum. Let M be an (R, R)-bimodule, then $\mathrm{THH}(R; M) = M \wedge^L_{R \wedge R^{\mathrm{op}}} R$.*

Proof. Replace R by $B(R, R, R) \simeq R$. Then $\mathrm{THH}(R; M)_n = M \wedge^L_{R\wedge R^{\mathrm{op}}} R \simeq M \wedge_{R \wedge R^{\mathrm{op}}} B(R, R, R)_n \simeq M \wedge_{R\wedge R^{\mathrm{op}}} R^{\wedge(n+2)} \simeq M \wedge R^{\wedge n}$. □

The definition from Lemma 4.2.11 is something reasonable to look at in algebra. We want to check that this is an extension of the algebraic theory, so restricting to algebra should recover Hochschild homology.

Example 4.2.12. Let R be an associative ring. Let M be an (R, R)-bimodule. Then $M \otimes_{R \otimes R^{\mathrm{op}}} R$ is given by

$$M \otimes R/(m(r \otimes s) \otimes u) \sim (m \otimes (r \otimes s)u).$$

Note that $m(r \otimes s) = smr$ and $(r \otimes s)u = rus$. The relation becomes $(m \otimes r) \sim (mr \otimes 1) \sim (rm \otimes 1)$. Thus $M \otimes_{R \otimes R^{\mathrm{op}}} R = M/(mr \sim rm)$. This recovers $\mathrm{HH}_0(R; M)$, and more generally recovers $\mathrm{HH}_*(R; M)$ via derived tensoring.

It seems a little strange that there is an apparent degree shift in the cyclic bar construction with $[n] \mapsto R^{\wedge(n+1)}$, but this has something to do with the simplicial structure for S^1. Consider again $S^1_\bullet$ and note that $\Delta[1] = \mathrm{Hom}_\Delta([n], [1])$. Combinatorially, using stars and bars, we know that $|\mathrm{Hom}_\Delta([n], [1])| = n + 2$. To get S^1_n we identify the endpoints, leaving $n + 1$ many simplices. This gives $(R \otimes S^1_\bullet)_n = R^{\wedge(n+1)}$.

Now we go through the most basic THH examples that we can.

Example 4.2.13. The simplest THH computation we can do is $\mathrm{THH}(\mathbb{S}) \simeq \mathbb{S}$. Since $\mathbb{S} \wedge \mathbb{S} \cong \mathbb{S}$, we get that $(\mathrm{THH}(\mathbb{S}))_n \cong \mathbb{S}^{\wedge(n+1)} \cong \mathbb{S}$.

Example 4.2.14 ([34, Cor. IV.3.3])**.** Let X be a based space, and consider the spectrum $\Sigma^\infty \Omega X$. Then $\mathrm{THH}(\Sigma^\infty \Omega X) \simeq \Sigma^\infty LX$, where LX is the free loop space of X from 2.1.8. This provides geometric intuition for why THH is naturally an S^1-spectrum, the cyclic structure encodes rotations of loops.

We will see in §4.3 that the next simplest computation is much less simple.

One of the key properties of THH is that it naturally forms a cyclotomic spectrum from Definition 3.3.5, meaning it carries extra compatibility between its S^1-action and its fixed point spectrum. This structure is central to topological cyclic homology later on.

Theorem 4.2.15. *Let R be a bounded-below, commutative ring spectrum. Then* $\mathrm{THH}(R)$ *admits canonical cyclotomic structure maps*

$$\varphi_p\colon\ \mathrm{THH}(R) \to \mathrm{THH}(R)^{tC_p}$$

for each prime p, which make $\mathrm{THH}(R)$ *a cyclotomic S^1–spectrum.*

Think of the cyclic bar construction as a spectral refinement of free loops. For a space X, $B^{\mathrm{cy}}(\Omega X) \simeq LX = \mathrm{Map}(S^1, X)$ and $\mathrm{THH}(\Sigma^\infty \Omega X) \simeq \Sigma^\infty LX$. Rotating loops gives the S^1–action. The extra "cyclotomic" structure encodes the operation of taking p-th roots of loops. The C_p–fixed points of LX identify with LX again since a loop invariant under rotating by $1/p$ is the p-th power of a unique loop, thus

$$\Phi^{C_p}\left(\Sigma^\infty_{C_p} LX\right) \simeq \Sigma^\infty (LX)^{C_p} \simeq \Sigma^\infty LX.$$

Cyclotomic structure abstracts this to $\mathrm{THH}(R)$. The Frobenius map φ_p is the spectral version of "take a p-th root of a loop."

To prove this theorem, we introduce the *Tate diagonal*. To motivate this Tate diagonal, we compare what it does to something in algebra. In algebra, for R a ring, one would like to use the diagonal $R \to R^{\wedge p}$, but it is not additive; its failure of additivity factors through transfers (norms) coming from the C_p–action. The Tate construction kills transfers: by definition X^{tC_p} is the cofiber of the splice $X_{hC_p} \xrightarrow{S} X^{hC_p}$, and on π_0 the corresponding map is the Tate splice $H_0^{\mathrm{grp}} \xrightarrow{S} H^0_{\mathrm{grp}}$. Hence, after applying $(-)^{tC_p}$ the diagonal becomes additive and multiplicative up to modding out by these transfers, which is exactly what is needed for the cyclic bar faces and degeneracies to commute with the construction. This explains conceptually why φ_p exists and behaves multiplicatively on $\mathrm{THH}(R)$.

Proof sketch. Consider the p–fold edgewise subdivision $sd_p B^{\mathrm{cy}}_\bullet(R)$, with

$$(sd_p B^{\mathrm{cy}}_\bullet(R))_n \simeq R^{\wedge(p(n+1))} \cong (R^{\wedge p})^{\wedge(n+1)}.$$

There is a canonical C_p–action on each level given by rotating the p blocks; this action is compatible with faces and degeneracies, hence $sd_p B^{\mathrm{cy}}_\bullet(R)$ is a cyclic object with a levelwise C_p–action. By Theorem 4.1.13, $|sd_p B^{\mathrm{cy}}_\bullet(R)| \simeq |B^{\mathrm{cy}}_\bullet(R)| = \mathrm{THH}(R)$.

There is a canonical Tate diagonal $\Delta_p \colon R \to (R^{\wedge p})^{tC_p}$ natural in R. For categorical reasons [34, Thm. I.3.1], Δ_p extends multiplicatively to all tensor powers:

$$\Delta_p^{(n+1)} \colon R^{\wedge(n+1)} \to \left((R^{\wedge p})^{\wedge(n+1)}\right)^{tC_p} \cong \left(R^{\wedge(p(n+1))}\right)^{tC_p}.$$

These maps assemble into a map of cyclic objects

$$B^{\mathrm{cy}}_\bullet(R) \to (sd_p B^{\mathrm{cy}}_\bullet(R))^{tC_p}$$

where compatibility with faces and degeneracies follows from the categorical reasons [34, Thm. I.3.1]. Geometrically realising this map gives

$$\varphi_p\colon \operatorname{THH}(R) \to |sd_p B^{\mathrm{cy}}_\bullet(R)|^{tC_p} \simeq \operatorname{THH}(R)^{tC_p},$$

the desired cyclotomic Frobenius φ_p. The residual S^1–action on $\operatorname{THH}(R)^{tC_p}$ is twisted by the p-th power map $\phi_p\colon S^1 \xrightarrow{(-)^p} S^1$ (coming from edgewise subdivision), yielding the cyclotomic compatibility required in the definition. □

There is a linearisation map going from THH to HH, which is useful for trace method approximations to algebraic K-theory.

Theorem 4.2.16 ([9, §5]). *If R is a connective ring spectrum, then there is a natural map $\pi_k \operatorname{THH}(R) \to \operatorname{HH}_k(\pi_0 R)$, which is an isomorphism for $k = 0$.*

4.3 Computing $\operatorname{THH}(\mathbb{F}_p)$

The next simplest computation with THH is $\operatorname{THH}(\mathbb{F}_p)$. This computation was originally done by Bökstedt, in his unpublished paper [8].

Theorem 4.3.1 ([8, Thm. 1.1]). *Let k be a perfect field of characteristic p, then $\pi_* \operatorname{THH}(k) \cong k[\mu]$ the free polynomial algebra on μ with $|\mu| = 2$.*

Taking the special case of $k = \mathbb{F}_p$, this topological version is much cleaner than $\operatorname{HH}_*(\mathbb{F}_p)$ as seen in Example 4.2.4.

There have been new ways to do this computation since it was originally done. We will present a few ways for the special case of $k = \mathbb{F}_p$, and in order to do the computation we will need many new tools.

Definition 4.3.2. The (mod p) *Steenrod algebra* is $\mathcal{A}_p := H\mathbb{F}_p^* H\mathbb{F}_p$. Dually, the (mod p) *dual Steenrod algebra* is $\mathcal{A}_* = H\mathbb{F}_{p*} H\mathbb{F}_p$.

The Steenrod algebra is the algebra of all stable cohomology operations in mod p cohomology, which we know is represented by $H\mathbb{F}_p$ by Example 2.2.24.

Theorem 4.3.3 ([32, §2]). *For odd primes p, the (mod p) Steenrod algebra $\mathcal{A}_p$ is the free associative graded algebra generated by β and P^i for $i \geq 0$ subject to the Adem relations. At $p = 2$, the algebra is generated by the Steenrod squares Sq^i for $i \geq 0$ subject to the Adem relations.*

Theorem 4.3.4 ([32, Lem. 6]). *The dual Steenrod algebra takes the following forms*

$$\mathcal{A}_* = \begin{cases} \mathbb{F}_2[\xi_1, \xi_2, ...] & p = 2,\ |\xi_n| = 2^n - 1 \\ \mathbb{F}_p[\xi_1, \xi_2, ...] \otimes E(\tau_0, \tau_1, ...) & p > 2,\ |\xi_n| = 2(p^n - 1),\ |\tau_n| = 2p^n - 1. \end{cases}$$

The importance of $\mathcal{A}_*$ is that it governs the $H\mathbb{F}_p$–(co)homology of smash powers of $H\mathbb{F}_p$. By Künneth (over a field) and the definitions in Example 2.2.24,

$$(H\mathbb{F}_p)_* \left(H\mathbb{F}_p^{\wedge(s+1)}\right) \cong \mathcal{A}_*^{\otimes(s+1)}.$$

Similarly, using that $\pi_*(H\mathbb{F}_p \wedge -) \cong (H\mathbb{F}_p)_*(-)$,

$$\pi_* \left(H\mathbb{F}_p^{\wedge(s+1)}\right) \ \cong\ (H\mathbb{F}_p)_* \left(H\mathbb{F}_p^{\wedge s}\right) \cong \mathcal{A}_*^{\otimes s}.$$

There are two common methods of computing THH($\mathbb{F}_p$), both of which involve a spectral sequence. The first method uses a spectral sequence on the homotopy groups, the second and original method due to Bökstedt [8] uses a spectral sequence on the homology groups. The homotopy method is simpler to compute, but harder to see the multiplicative structure. On the other hand, we can see the multiplicative structure using homology and *Dyer-Lashof homology operations* (to be partially defined in Definition 4.3.5). We will give a high level of both methods of computation.

Both constructions begin with the natural skeletal filtration of the simplicial spectrum defining THH. For any homology theory $E_*(-)$ and commutative ring spectrum A, this filtration yields a spectral sequence

$$E^1_{s,t} = E_t\left(A^{\wedge(s+1)}\right) \implies E_{s+t}\,\mathrm{THH}(A).$$

In particular, for E_* satisfying the Künneth formula (such as $H\mathbb{F}_p$) we get (by using the formula) that the E^1-term becomes tensor powers of $E_*(A)$. That is,

$$E^1_{s,t} = E_t(A)^{\otimes_{E_*(*)}(s+1)}.$$

To illustrate the structure, recall that the Tor groups of simple algebras can be computed from projective resolutions. Let k be a ring, then we compute $\mathrm{Tor}_*^{k[x]}(k,k)$. A projective resolution of k is

$$0 \leftarrow k \leftarrow k[x] \xleftarrow{\cdot x} k[x] \leftarrow 0$$

tensoring with k and taking homology, we get

$$0 \leftarrow k \xleftarrow{x=0} k \leftarrow 0.$$

We conclude that

$$\mathrm{Tor}_s^{k[x]}(k,k) = \begin{cases} k & s = 0,1 \\ 0 & \text{else.} \end{cases}$$

Similarly, we compute $\mathrm{Tor}_*^{k[x]/x^2}(k,k)$:

$$0 \leftarrow k \xleftarrow{\cdot x} k[x]/x^2 \xleftarrow{\cdot x} k[x]/x^2 \leftarrow \cdots$$

tensoring with k and taking homology, we get

$$0 \leftarrow k \xleftarrow{0} k \xleftarrow{0} k \leftarrow \cdots$$

since $k[x]/x^2 \otimes_{k[x]/x^2} k \cong k$. We conclude that

$$\mathrm{Tor}_s^{k[x]/x^2}(k,k) \cong \begin{cases} k & s \geq 0 \\ 0 & \text{else.} \end{cases}$$

With some more work, we find that this has the ring structure of the *divided power algebra* $\Gamma(k)$, see [12, Ch XI]. The divided power algebra $\Gamma(y)$ is generated as a k-module by $\gamma_i(y)$ with $\gamma_i(y)\gamma_j(y) = \binom{i+j}{i}\gamma_{i+j}(y)$.

4.3.1 Homotopy Computation

Let our homology theory be π_*^s and $A = H\mathbb{F}_p$. Then $E^1_{s,t} = \pi_t(H\mathbb{F}_p^{\wedge(s+1)}) \cong \mathcal{A}_*^{\otimes s}$, so we get the below projective resolution (associated chain complex of the simplicial abelian group)

$$\mathbb{F}_p \Longrightarrow \mathcal{A}_* \Rrightarrow \mathcal{A}_*^{\otimes 2} \cdots$$

We recognise the homology of this resolution as $\mathrm{Tor}_{*,*}^{\mathcal{A}_*}(\mathbb{F}_p,\mathbb{F}_p)$. Thus, we get

$$E^2_{*,*} = E(\sigma\xi_1, \sigma\xi_2, \ldots) \otimes \mathcal{A}_*, \qquad (p=2),$$
$$E^2_{*,*} = \Gamma(\sigma\tau_0, \sigma\tau_1, \ldots) \otimes E(\sigma\xi_1, \sigma\xi_2, ...) \otimes \mathcal{A}_* \quad (p \text{ odd}),$$

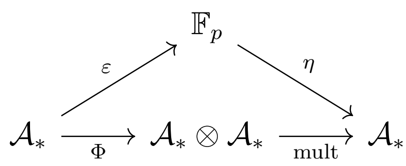
where $\sigma \in \pi_1(S^1)$ is the fundamental class. In the odd p case, there are differentials to consider, which we discuss in the homology computation. There are also multiplicative extensions which are not easily solved in the homotopy case.

4.3.2 Homology Computation

Let our homology theory be $H\mathbb{F}_p$ and $A = H\mathbb{F}_p$. Then $E^1_{s,t} = H\mathbb{F}_p(H\mathbb{F}_p^{\wedge(s+1)}) \cong \mathcal{A}_*^{\otimes(s+1)}$, so we get the below projective resolution (associated chain complex of the simplicial abelian group)

$$\mathcal{A}_* \;\Longleftarrow\Longrightarrow\; \mathcal{A}_*^{\otimes 2} \;\Longleftarrow\Longrightarrow\; \mathcal{A}_*^{\otimes 3} \;\Longleftarrow\Longrightarrow\; \cdots$$

We recognise the homology of this as $\mathrm{HH}_*(\mathcal{A}_*)$ from Definition 4.2.1 which by Lemma 4.2.3, is $\mathrm{Tor}^{\mathcal{A}_*\otimes\mathcal{A}_*^{\mathrm{op}}}_{*,*}(\mathcal{A}_*, \mathcal{A}_*)$. Since $\mathcal{A}_*$ is graded commutative, $\mathcal{A}_* = \mathcal{A}_*^{\mathrm{op}}$. We can then do a change of basis $\Phi\colon \mathcal{A}_* \to \mathcal{A}_* \otimes \mathcal{A}_*$ given by $\tau_i \mapsto \tau_i \otimes 1 - 1 \otimes \tau_i$ and $\xi_j \mapsto \xi_j \otimes 1 - 1 \otimes \xi_j$, as in the below diagram.



These basis elements act trivially on $\mathcal{A}_*$, so this Tor simplifies to $\mathrm{Tor}^{\mathcal{A}_*}_{*,*}(\mathbb{F}_p, \mathbb{F}_p) \otimes \mathcal{A}_*$.

Similar to before, we get

$$\begin{aligned} E^2_{*,*} &= E(\sigma\xi_1, \sigma\xi_2, \ldots) \otimes \mathcal{A}_*, && (p = 2),\\ E^2_{*,*} &= \Gamma(\sigma\tau_0, \sigma\tau_1, \ldots) \otimes E(\sigma\xi_1, \sigma\xi_2, \ldots) \otimes \mathcal{A}_* && (p \text{ odd}). \end{aligned}$$

where as before, $\sigma \in \pi_1(S^1)$ is the fundamental class. We can then use the so-called *Dyer-Lashof* homology operations to show that $(\sigma\xi_i)^2 = \sigma\xi_{i+1}$ [11, Ch III §2.2, 2.3].

Definition 4.3.5. Let X be a commutative ring spectrum. Then in $H\mathbb{F}_p$ homology of X, we have *Dyer-Lashof* homology operations,

$$\begin{aligned} Q^s :& H_n(X; \mathbb{F}_2) \longrightarrow H_{n+s}(X; \mathbb{F}_2) && (p = 2),\\ Q^s :& H_n(X; \mathbb{F}_p) \longrightarrow H_{n+2s(p-1)}(X; \mathbb{F}_p) && (p \text{ odd}), \end{aligned}$$

which satisfy $Q^{|x|}(x) = x^p$.

In our case, it turns out that $Q^{p^n}(\xi_n) = \xi_{n+1}$ and Q^s commutes with σ [11, Ch III §2.2, 2.3].

In the $p = 2$ case, there are no possible differentials for degree reasons, so we have $H_*(\mathrm{THH}(\mathbb{F}_2); \mathbb{F}_2) \cong \mathcal{A}_* \otimes \mathbb{F}_2[\mu]$. In the odd p case, there are nontrivial differentials $d_{p-1}(\gamma_n(\sigma\tau_i)) = \sigma\xi_{i+1}\gamma_{n-p}(\sigma\tau_i)$ [11, Ch III §2.2, 2.3]. The spectral sequence collapses on the $E^p = E^\infty$ page with $E^p_{*,*} = \mathcal{A}_* \otimes P_p(\sigma\tau_0, \sigma\tau_1, ...)$, the truncated polynomials in $\mathbb{F}_p$. Using the Dyer-Lashof operations, we get $(\sigma\xi_i)^p = \sigma\xi_{i+1}$. Thus the odd–primary Bökstedt $E^p = E^\infty$ page gives

$$H_*(\mathrm{THH}(\mathbb{F}_p); \mathbb{F}_p) \cong \mathcal{A}_* \otimes \mathbb{F}_p[\mu], \quad |\mu| = 2.$$

Finally, a small Adams spectral sequence computation gives $\pi_* \mathrm{THH}(\mathbb{F}_p) \cong \mathbb{F}_p[\mu]$, $|\mu| = 2$.

4.3.3 Thom computation

Another perspective on $\mathrm{THH}(\mathbb{F}_p)$ comes from its identification as a Thom spectrum. Thom spectra provide a bridge between homotopy theory and vector bundle theory, and their structure makes them particularly well suited for calculations in THH.

Let $E \to X$ be a rank n vector bundle over the base space X. The Thom space of E is defined as $Th(E) \cong D(E)/S(E)$, where $D(E)$ is the unit disk bundle of E and $S(E)$ is the unit sphere bundle. Equivalently, if we take the fiberwise one point compactification then $Th(E) \cong S^E/s_\infty(X)$ where S^E is an S^n fiber bundle by taking the one point compactification of each fiber, and $s_\infty(X)$ is the section at infinity that identifies these extra points.

Example 4.3.6. Consider the trivial bundle $X \times \mathbb{R}^n \to X$, we have that $Th(X \times \mathbb{R}^n \to X) = \Sigma^n(X_+)$.

We can do something analogous for spectra. A *Thom spectrum* is determined by a map $f\colon X \to BGL_1(\mathbb{S})$ classifying a virtual spherical fibration. This is essentially looking at maps with virtual degree 0 (so no shifts needed in formal differences). Concretely, if $f_n\colon X \to BO(n)$ is a representative in finite dimensions, then the n-th level of the associated Thom spectrum is $Mf(n) = Th(f_n^*\gamma_n)$ where γ_n is the universal bundle over $BO(n)$. The structure maps $\Sigma Mf(n) \to Mf(n{+}1)$ arise from the inclusions $\mathbb{R}^n \to \mathbb{R}^{n+1}$.

As a special case of the Hopkins–Mahowald theorem, we have that $H\mathbb{F}_p$ is a Thom spectrum.

Theorem 4.3.7 (Hopkins–Mahowald, [3, §8])**.** *The map $f\colon \Omega^2 S^3 \to BGL_1(\mathbb{S})$ exists and classifies the element in $\pi_1(BGL_1(\mathbb{S})) \cong \mathbb{Z}$, with Thom spectrum Mf equivalent to $H\mathbb{F}_p$. In particular, $Mf \simeq H\mathbb{F}_p$ as ring spectra.*

Moreover, the f is mod p oriented, so the Thom isomorphism gives an isomorphism on homology $H_*(Mf;\mathbb{F}_p) \cong H_*(\Omega^2 S^3;\mathbb{F}_p)$ with no degree shift (the virtual bundle classified by f has degree 0).

We give motivation for the theorem by showing this homology result.

Theorem 4.3.8. *The dual Steenrod algebra can be identified with the homology of* $\Omega^2 S^3$. *That is,*

$$\mathcal{A}_* = H\mathbb{F}_{p*}H\mathbb{F}_p \cong H_*(\Omega^2 S^3;\mathbb{F}_p).$$

Proof. We use the Serre spectral sequence on the pathspace fibration $\Omega S^3 \to PS^3 \to S^3$. Since PS^3 is contractible, everything in the spectral sequence must die, so we can do some investigative work to compute $H_*(\Omega S^3;\mathbb{F}_p)$ from $H_*(S^3;\mathbb{F}_p)$. The only possible nonzero d_r out of the bottom row is the transgression $d_3\colon E^3_{3,0} \to E^3_{0,2}$. Let $u \in H_3(S^3;\mathbb{F}_p)$ be the fundamental class and define $y \coloneqq d_3(u) \in H_2(\Omega S^2;\mathbb{F}_p)$. Then, by multiplicativity and the vanishing of all positive degrees in the target, we find that

$$H_*(\Omega S^3;\mathbb{F}_p) \cong \mathbb{F}_p[y], \quad |y| = 2.$$

Similarly, we use the Serre spectral sequence on the pathspace fibration $\Omega^2 S^3 \to P\Omega S^3 \to \Omega S^3$. The first possible nonzero differential out of the base's degree 2 generator y is $d_2\colon E^2_{2,0} \to E^2_{0,1}$. Since the total must die in positive degrees, we must have $d_2(y) \neq 0$. Define the transgression $a_0 \coloneqq d_2(y) \in H_1(\Omega^2 S^3;\mathbb{F}_p)$. By the Leibniz rule, $d_2(y^m) = my^{m-1}x$. Note that $d_2(y^p) = py^{p-1}a_0 = 0 \mod p$, so y^p survives to E^3. Again, since the total space is contractible, every surviving positive degree class must eventually be hit by a higher differential. The first page on which $y^{p^k} \in E^r_{2p^k,0}$ can be killed is $r = 2p^k$, by $d_{2p^k}\colon E^{2p^k}_{2p^k,0} \to E^{2p^k}_{0,2p^k-1}$. This forces new transgression classes $a_k \coloneqq d_{2p^k}(y^{p^k}) \in H_{2p^k-1}(\Omega^2 S^3;\mathbb{F}_p)$.

For $p = 2$, all signs vanish and the odd degree generators can freely multiply. We find

$$H_*(\Omega^2 S^3;\mathbb{F}_2) \cong \mathbb{F}_2[a_0, a_1, ...], \quad |a_k| = 2^k - 1.$$

For p odd, we need to remember graded commutativity and the even degree polynomial generators b_k in degrees $2p^k$, created to let the d_{2p^k} hit y^{p^k}. We find

$$H_*(\Omega^2 S^3;\mathbb{F}_p) \cong E(a_0, a_1, ...) \otimes \mathbb{F}_p[b_1, b_2, ...], \quad |a_k| = 2p^k - 1,\ |b_k| = 2p^k.$$

Combining all these facts gives

$$H\mathbb{F}_{p*}H\mathbb{F}_p = H_*(H\mathbb{F}_p;\mathbb{F}_p) \cong H_*(\Omega^2 S^3;\mathbb{F}_p),$$

as required. □

With this new perspective, we give a formula for computing THH of a Thom spectrum.

Theorem 4.3.9 ([4, Thm. 1.4])**.** *Let $f\colon X \to BGL_1(\mathbb{S})$ and Mf its Thom spectrum, then there is an equivalence*

$$\mathrm{THH}(Mf) \simeq Mf \wedge (BX)_+$$

where BX is the deloop space *of X, such that $X \simeq \Omega BX$.*

Applying this formula to $H\mathbb{F}_p$ gives an alternate way of computing $\mathrm{THH}(\mathbb{F}_p)$:

$$\mathrm{THH}(\mathbb{F}_p) \simeq H\mathbb{F}_p \wedge (B\Omega^2 S^3)_+ \simeq H\mathbb{F}_p \wedge (\Omega S^3)_+ \simeq H\mathbb{F}_{p*}(\Omega S^3) \cong \mathbb{F}_p[\mu], \quad |\mu| = 2.$$

We note that this computation was significantly easier than the one in the previous subsection 4.3.2. For this reason, it is an important piece of research to determine which spectra can be realised as Thom spectra in order to compute their THH.

4.4 Topological Cyclic Homology

While THH is already a powerful invariant, the true significance comes from the connection to algebraic K-theory. Topological cyclic homology was introduced by Bökstedt, Hsiang and Madsen [9] as a tractable approximation to algebraic K-theory, and has since been refined by Nikolaus and Scholze [34] into a very elegant theory.

Just as we had old and new definitions of cyclotomic, we will give the old and new definitions of topological cyclic homology.

Definition 4.4.1 ([9, Def. 5.12])**.** Let A be a ring spectrum and fix a prime p. Consider the family of fixed points $\mathrm{THH}(A)^{C_{p^n}}$ for all n. There are two maps, *restriction* and *Frobenius* $R, F\colon \mathrm{THH}(A)^{C_{p^n}} \to \mathrm{THH}(A)^{C_{p^{n-1}}}$, the restriction and the Frobenius. The Frobenius map is the inclusion of fixed points along the subgroup $C_{p^{n-1}} \subset C_{p^n}$. The restriction map is the composite $(\varphi_p)^{C_{p^{n-1}}} \circ \mathrm{can}$, where $\mathrm{can}\colon \mathrm{THH}(A)^{C_{p^n}} \to \left(\Phi^{C_p}\,\mathrm{THH}(A)\right)^{C_{p^{n-1}}}$ is the *canonical* map from categorical fixed points to geometric fixed points, and $\varphi_p\colon \Phi^{C_p}\,\mathrm{THH}(A) \to \mathrm{THH}(A)$ is the (old) cyclotomic map from Definition 3.3.4. The *topological cyclic homology of A at p* is defined as the homotopy limit

$$\mathrm{TC}(A;p) := \underset{R,F}{\underleftarrow{\mathrm{holim}}} \left(\mathrm{THH}(A)^{C_{p^n}} \to \mathrm{THH}(A)^{C_{p^{n-1}}}\right).$$

It may seem like the restriction and Frobenius maps have been named incorrectly, but this is standard in the literature and done in order to match up with the Witt vectors in §4.5.

This version of TC has the advantage of being conceptually close to an algebraic analogue (cyclic homology), but it is technically demanding to compute.

Nikolaus and Scholze redefine TC by removing (genuine) equivariant homotopy theory from the definition.

Definition 4.4.2 ([34, Def. II.1.8/Prop. II.1.9])**.** Let A be a connective ring spectrum. The *negative topological cyclic homology* of A is

$$\mathrm{TC}^-(A) \coloneqq \mathrm{THH}(A)^{hS^1}.$$

The *periodic topological cyclic homology* of A is

$$\mathrm{TP}(A) \coloneqq (\mathrm{THH}(A)^{tC_p})^{hS^1/C_p}.$$

The *topological cyclic homology* of A is given by the homotopy equaliser

$$\mathrm{TC}(A) \coloneqq \mathrm{hoeq}\left(\mathrm{TC}^-(A) \underset{\prod_p \varphi_p^{hS^1}}{\overset{\mathrm{can}}{\rightrightarrows}} \prod_p \mathrm{TP}(A) \right).$$

We can consider TC one prime at a time with

$$\mathrm{TC}(A;p) = \mathrm{hoeq}\left(\mathrm{THH}(A)^{hS^1} \underset{\varphi_p^{hS^1}}{\overset{\mathrm{can}}{\rightrightarrows}} \left(\mathrm{THH}(A)^{tC_p}\right)^{hS^1} \right).$$

If we only care up to p-completion, then note $\mathrm{TP}(A)^\wedge_p \simeq \mathrm{THH}(A)^{tS^1}$ and

$$\mathrm{TC}(A;p) = \mathrm{hoeq}\left(\mathrm{THH}(A)^{hS^1} \underset{\varphi_p^{hS^1}}{\overset{\mathrm{can}}{\rightrightarrows}} \mathrm{THH}(A)^{tS^1} \right).$$

We can instead consider $\mathrm{THH}(A)$ as only p-cyclotomic, and define

$$\mathrm{TC}(A;p) = \mathrm{hoeq}\left(\mathrm{THH}(A)^{hC_{p^\infty}} \underset{\varphi_p^{hC_{p^\infty}}}{\overset{\mathrm{can}}{\rightrightarrows}} \left(\mathrm{THH}(A)^{tC_p}\right)^{hC_{p^\infty}} \right).$$

This gives the p-complete version

$$\mathrm{TC}(A;p)^\wedge_p = \mathrm{hoeq}\left(\mathrm{THH}(A)^{hC_{p^\infty}} \underset{\varphi_p^{hC_{p^\infty}}}{\overset{\mathrm{can}}{\rightrightarrows}} \mathrm{THH}(A)^{tC_{p^\infty}} \right).$$

These definitions of topological cyclic homology have the virtue of requiring only a single S^1 or C_{p^∞}-action, and they extend more naturally to higher-categorical settings. We will go through two examples of computing TC once we introduce the correct tools to do so – some more spectral sequences.

4.4.1 Spectral Sequence Detour

We aim to build spectral sequences for the spectra seen in §3.3. If $Y \simeq \operatorname{holim} Y_s$ is the homotopy limit of a tower of fibrations

$$\cdots \to Y_s \xrightarrow{p_s} Y_{s-1} \to \cdots \to Y_0$$

write F_s for the fiber of p_s. The *Bousfield-Kan* spectral sequence (BKSS) associated to this tower has signature

$$E_1^{s,t} = \pi_{t-s}(F_s) \implies \pi_{t-s}(Y)$$

with d_1 induced by the connecting maps of the fiber sequences [10].

We apply this construction to towers coming from skeletal filtrations of EG_+.

Theorem 4.4.3. *Let X be a G-spectrum, then we have the homotopy fixed point, homotopy orbit and Tate spectral sequences, given by the below signatures*

- *Homotopy fixed point:* $E_2^{s,t} = H^s_{\mathrm{grp}}(G; \pi_t X) \implies \pi_{t-s}(X^{hG})$,
- *Homotopy orbit:* $E^2_{s,t} = H_s^{\mathrm{grp}}(G; \pi_t X) \implies \pi_{s+t}(X_{hG})$,
- *Tate:* $\widehat{E}_2^{s,t} = \widehat{H}^s_{\mathrm{grp}}(G; \pi_t X) \implies \pi_{t-s}(X^{tG})$.

The first two arise from the skeletal filtration of EG, the Tate sequence arises from the same filtration applied to the norm cofiber definition of X^{tG}.

Proof sketch. Take $EG = |B_\bullet(G, G, *)|$, the two sided bar construction from Remark 4.2.8. Then, EG is a free, contractible G–CW complex and $BG = EG/G$. Let $sk_t EG$ denote the t–skeleton and set $K_t := sk_t EG/sk_{t-1}EG \simeq \bigvee_{\{t\text{–cells of } EG\}} S^t$, on which G permutes summands freely.

We start with homotopy fixed points. Use $X^{hG} \simeq M(EG_+, X)^G$ and form the tower

$$\cdots \to M(sk_t EG_+, X)^G \longrightarrow M(sk_{t-1}EG_+, X)^G \to \cdots \to M(S^0, X)^G.$$

Each map is a fibration and its fiber is

$$\begin{aligned} \mathrm{Fib}\Big(M(sk_t EG_+, X) \to M(sk_{t-1}EG_+, X)\Big)^G &\simeq M(K_t, X)^G \\ &\simeq \left(\prod_{\{t\text{–cells of } EG\}} \Sigma^{-t} X \right)^G . \end{aligned}$$

Since G acts freely on t–cells of EG, taking G–categorical fixed points identifies this product with the product indexed over t-cells of BG. Therefore, $E_1^{s,t} \cong C^s_{\text{cell}}(BG;\pi_t X)$, the cellular cochains of BG with *local coefficients* in the G–module $\pi_t X$. Here the coefficient system is the constant local system determined by the G–action on $\pi_t X$ (coming from the G–action on X). The cellular chains of the universal cover EG acquire a G–action, and the cochains with coefficients in the G–module $\pi_t X$ are the G–equivariant maps $C_s(EG) \to \pi_t X$. The d_1 is induced by cellular attaching maps, hence computes group cohomology, giving $E_2^{s,t} \cong H^t_{\text{grp}}(G;\pi_s X)$.

For homotopy orbits the argument is similar. Use $X_{hG} \simeq (EG_+ \wedge X)/G$ and the filtration by $sk_t EG_+ \wedge X$. Successive cofibers are wedges $\bigvee_{\{t\text{–cells of } BG\}} \Sigma^t X$, so $E^1_{s,t} \cong C_s^{\text{cell}}(BG;\pi_t X)$ and $E^2_{s,t} \cong H_s(G;\pi_t X)$, with target $\pi_{s+t}(X_{hG})$.

For the Tate construction, recall that we defined X^{tG} as the cofiber of the splice $S\colon X_{hG} \to X^{hG}$. Filtering both sides by skeleta and passing to the cofiber yields a BKSS whose E_1 is the so-called *cellular Tate complex*, so $E_2^{s,t} = \widehat{H}^s_{\text{grp}}(G;\pi_t X)$ and the target is $\pi_{t-s} X^{tG}$.

Finally, for the towers built from a finite (or locally finite) CW filtration of EG, the BKSS is strongly convergent under standard hypotheses; for instance, if the π_* of the successive fibers are degreewise finite type. In particular, for bounded-below G–spectra X with $\pi_t X$ of finite type, the spectral sequences above converge strongly to the indicated targets (see [10] for general convergence criteria). □

Note that any simplicial or CW model of EG gives the same spectral sequences. The grading places the group (co)homology in the t direction and internal homotopy in the s direction, with total degree $t - s$ for cohomological sequences and $t + s$ for the homological one.

4.4.2 Computing TC

Since the easiest THH computation was of the sphere spectrum, it would be natural to look at TC($\mathbb{S}$). However, it turns out that this computation is very difficult.

Theorem 4.4.4 ([9, Eq. 9.1])**.**

$$\begin{aligned} \mathrm{TC}(\mathbb{S};p)^\wedge_p &\simeq \mathrm{hoeq}\left(M(\mathbb{C}P^\infty,\mathbb{S}) \underset{\varphi_p}{\overset{\mathrm{can}}{\rightrightarrows}} M(\mathbb{C}P^\infty,\mathbb{S}^\wedge_p) \right) \\ &\simeq \mathbb{S}^\wedge_p \vee \mathrm{hofib}\big(\Sigma\left(\Sigma^\infty \mathbb{C}P^\infty\right) \to \mathbb{S}\big)^\wedge_p. \end{aligned}$$

Instead, we do the more tractable computation for $\mathrm{TC}(\mathbb{F}_p; p)$.

Theorem 4.4.5.

$$\pi_n \mathrm{TC}(\mathbb{F}_p; p) \cong \begin{cases} \mathbb{Z}_p & n = 0, -1 \\ 0 & \textit{else.} \end{cases}$$

Proof. To start, note that $H\mathbb{F}_p$ is p-complete, so we do not run into any p-completion issues and can use the p-complete definition of TC. Now, recall from Theorem 4.3.1 that $\pi_* \mathrm{THH}(\mathbb{F}_p) \cong \mathbb{F}_p[\mu]$ with $|\mu| = 2$. Since the homotopy groups are concentrated in even degrees, multiplying by the fundamental class $\sigma \in \pi_1(S^1)$ is trivial, as it increases degree by 1. To compute $\mathrm{TC}(\mathbb{F}_p)$, we use all of the tools that we now have, combined with some results that we quote below, extending the computations from Example 3.3.12. We use the homotopy fixed point and Tate spectral sequences,

$$E_2^{*,*}(\mathrm{TC}^-(\mathbb{F}_p)) \cong H^*_{\mathrm{grp}}(S^1; \mathbb{F}_p) \otimes \mathbb{F}_p[\mu] \cong \mathbb{F}_p[t] \otimes \mathbb{F}_p[\mu] \implies \pi_* \mathrm{TC}^-(\mathbb{F}_p)$$
$$\widehat{E}_2^{*,*}(\mathrm{TP}(\mathbb{F}_p)) \cong \widehat{H}^*_{\mathrm{grp}}(S^1; \mathbb{F}_p) \otimes \mathbb{F}_p[\mu] \cong \mathbb{F}_p[t, t^{-1}] \otimes \mathbb{F}_p[\mu] \implies \pi_* \mathrm{TP}(\mathbb{F}_p)$$

with $|t| = (2, 0)$ from $H^2_{\mathrm{grp}}(S^1; \mathbb{F}_p)$ and $|\mu| = (0, 2)$ from $\pi_2 \mathrm{THH}(\mathbb{F}_p)$. All classes have even total degree, so both spectral sequences collapse at the E_2 page for degree reasons. We claim that there are multiplicative extensions, with p represented by $t\mu$ in $\pi_0 \mathrm{TC}^-(\mathbb{F}_p)$.

Consider the cofiber sequence $\mathrm{THH}(\mathbb{F}_p) \xrightarrow{p} \mathrm{THH}(\mathbb{F}_p) \to \mathrm{THH}(\mathbb{F}_p)/p$ and apply $(-)^{hS^1}$. This gives a long exact sequence

$$\cdots \to \pi_1(\mathrm{THH}(\mathbb{F}_p)/p)^{hS^1} \xrightarrow{\beta} \pi_0 \mathrm{THH}(\mathbb{F}_p)^{hS^1} \xrightarrow{\cdot p} \pi_0 \mathrm{THH}(\mathbb{F}_p)^{hS^1} \to \cdots$$

where β is the Bockstein connecting map. Morally, our claim is that μ is $\sigma\beta$. Concretely, in the S^1 homotopy fixed point spectral sequence for $\mathrm{THH}(\mathbb{F}_p)/p$, let $\tau_0 \in E_2^{1,0}$ denote the mod p Bockstein class. The generator $\sigma \in \pi_1(S^1)$ acts by rotation, on the spectral sequence this gives the d_2 differential $d_2(\tau_0) = t\sigma\tau_0 = t\mu$. Passing through the long exact sequence above, $d_2(\tau_0)$ identifies the boundary of multiplication by p with multiplication by t on the transgressed class, which exactly gives $p = t\mu$ in $\pi_0 \mathrm{THH}(\mathbb{F}_p)^{hS^1} = \pi_0 \mathrm{TC}^-(\mathbb{F}_p)$. We get a similar result for the Tate spectral sequence. A visual representation of these spectral sequences can be seen below in Figures 4.4.2 and 4.4.2. By multiplicativity of the spectral sequences, we get

$$\pi_{2m} \mathrm{TC}^-(\mathbb{F}_p) \cong \begin{cases} \mathbb{Z}_p\{\mu^m\} & \text{for } m \geq 0 \\ \mathbb{Z}_p\{t^{-m}\} & \text{for } m \leq 0, \end{cases} \quad \text{and} \quad \pi_{2m} \mathrm{TP}(\mathbb{F}_p) \cong \mathbb{Z}_p\{t^{-m}\}.$$

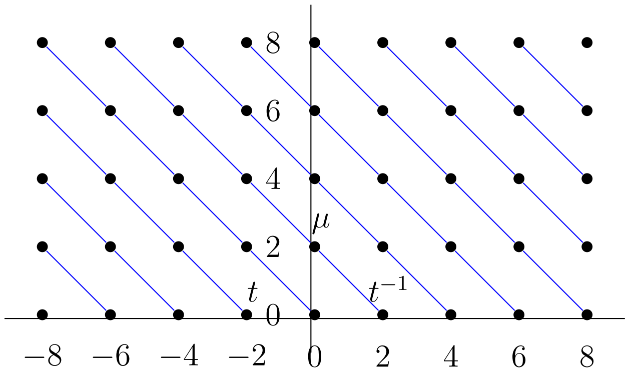


Figure 4.1: Tate $\widehat{E}_2 = \widehat{E}_\infty$ page.

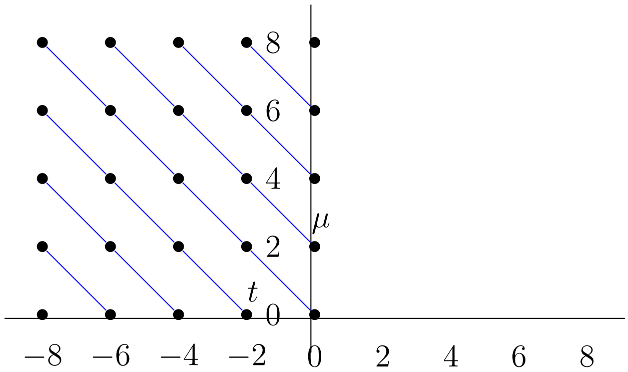


Figure 4.2: Homotopy fixed point $E_2 = E_\infty$ page.

Next, the canonical map $\mathrm{can}\colon \mathrm{TC}^-(\mathbb{F}_p) \to \mathrm{TP}(\mathbb{F}_p)$ is induced by the inclusion $hS^1 \to tS^1$, so is inclusion on the E_2–pages. In particular since $p = t\mu$, $\mathrm{can}(t) = t$, $\mathrm{can}(\mu) = pt^{-1}$. The Frobenius $\varphi_p\colon \mathrm{TC}^-(\mathbb{F}_p) \to \mathrm{TP}(\mathbb{F}_p)$ is a ring map with $\varphi_p(t) = pt$, $\varphi_p(\mu) = t^{-1}$ [34, Cor. IV.4.13]. To compute TC, we now use the cyclotomic fiber sequence

$$\mathrm{TC}(\mathbb{F}_p; p) \longrightarrow \mathrm{TC}^-(\mathbb{F}_p) \xrightarrow{\mathrm{can}-\varphi_p} \mathrm{TP}(\mathbb{F}_p)$$

and its long exact sequence on homotopy. We compute $\mathrm{can} - \varphi_p$ in cases.

For $m \leq -1$, we have $(\mathrm{can} - \varphi_p)(t^m) = t^m - (pt)^m = (1 - p^m)t^m$. Then since $1-p^m \in \mathbb{Z}_p^\times$, this is an isomorphism. Hence $\pi_{2m} \mathrm{TC}(\mathbb{F}_p; p) = \pi_{2m-1} \mathrm{TC}(\mathbb{F}_p; p) = 0$. Similarly, for $m \geq 1$, we have $(\mathrm{can} - \varphi_p)(\mu^m) = (p^m - 1)t^{-m}$. As before, since $p^m - 1 \in \mathbb{Z}_p^\times$, this is an isomorphism. Hence $\pi_{2m} \mathrm{TC}(\mathbb{F}_p; p) = \pi_{2m-1} \mathrm{TC}(\mathbb{F}_p; p) = 0$.

For $m = 0$, we have $(\mathrm{can} - \varphi_p)(1) = 1 - 1 = 0$. Exactness gives a surjection $\pi_0 \mathrm{TC}(\mathbb{F}_p; p) \twoheadrightarrow \pi_0 \mathrm{TC}^-(\mathbb{F}_p) \cong \mathbb{Z}_p$ and, since the map to $\pi_0 \mathrm{TP}(\mathbb{F}_p)$ is zero, an injection $\pi_{-1}\operatorname{TC}(\mathbb{F}_p;p) \hookrightarrow \pi_0 \operatorname{TC}^-(\mathbb{F}_p) \cong \mathbb{Z}_p$. Exactness at $\pi_0 \operatorname{TC}^-$ shows $\pi_{-1}\operatorname{TC}$ is the kernel of $\mathbb{Z}_p \to \mathbb{Z}_p$. Thus $\pi_0 \operatorname{TC}(\mathbb{F}_p;p) \cong \mathbb{Z}_p$ and $\pi_{-1}\operatorname{TC}(\mathbb{F}_p;p) \cong \mathbb{Z}_p$.

Combining the three cases proves

$$\pi_n \operatorname{TC}(\mathbb{F}_p;p) \cong \begin{cases} \mathbb{Z}_p & n = 0, -1, \\ 0 & \text{else}, \end{cases}$$

as claimed. □

Classically, the algebraic K-theory computation due to Quillen [35] has

$$K_n(\mathbb{F}_p) \cong \begin{cases} \mathbb{Z}/(p^i-1) & n = 2i-1 > 0, \\ \mathbb{Z} & n = 0, \\ 0 & n = 2i > 0, \end{cases}$$

so since $\mathbb{Z}/(p^i-1)$ has order prime to p, after p-completion we have

$$K_n(\mathbb{F}_p)^\wedge_p \cong \begin{cases} \mathbb{Z}_p & n = 0, \\ 0 & n \neq 0. \end{cases}$$

Note that $K(\mathbb{F}_p)$ is connective, so the cyclotomic trace $K(\mathbb{F}_p)^\wedge_p \to \operatorname{TC}(\mathbb{F}_p;p)$ is an isomorphism on π_0 and trivial in positive degrees, while TC has an additional $\mathbb{Z}_p$ in degree -1 coming from the fiber description. By the Dundas–Goodwillie–McCarthy theorem [14], we are able to 'bootstrap' this algebraic K-theory approximation from $\mathbb{F}_p$ to many other rings.

4.5 Relation to Witt Vectors

4.5.1 Witt Vectors

Witt vectors package "p-power data" of a ring A into a characteristic 0 object that lifts A modulo p. Concretely, for $A = \mathbb{F}_p$, we recover the p-adics $\mathbb{Z}_p$, so Witt vectors extend this to more general bases. To motivate the definition, we think of p-adics as a power series in p. Let $x \in \mathbb{Z}_p$, then we can write

$$x = \sum_{i=0}^{\infty} a_i p^i, \quad a_i \in [0, p-1].$$

The definition of Witt vectors generalises this.

Definition 4.5.1. Let p be prime and A a commutative ring. For $n \geq 1$, the *p-typical length n Witt vectors* form a set $W_n(A;p) = A^n$ with Witt coordinates $(a_0, ..., a_{n-1})$. Define the *Witt polynomials* $w_k := \sum_{j=0}^{k} p^j a_j^{p^{j-k}}$. Then the ring structure on $W_n(A;p)$ is defined so that the *ghost map*

$$\begin{aligned} w\colon W_n(A;p) &\to A^n \\ (a_0, \ldots, a_{n-1}) &\mapsto \langle w_0, \ldots, w_{n-1} \rangle \end{aligned}$$

is a natural ring map.

Addition and multiplication in $W_n(A;p)$ are uniquely determined by requiring that $w(x+y) = w(x)+w(y)$ and $w(xy) = w(x)w(y)$ coordinatewise in A^n. When p is clear from context or not important, we omit it from the notation.

There is a multiplicative (but not additive) section $[\cdot]\colon A \to W_n(A)$, the *multiplicative lift* given by $[a] = (a, 0, ..., 0)$. Then, there are three natural endomorphisms relating lengths n and $n-1$:

- Restriction: $R\colon W_n(A) \to W_{n-1}(A)$ by $R(a_0, ..., a_{n-1}) = (a_0, ..., a_{n-2})$.
- Verschiebung: $V\colon W_{n-1}(A) \to W_n(A)$ by $V(a_0, ..., a_{n-2}) = (0, a_0, ..., a_{n-2})$. It is additive (but not multiplicative).
- Frobenius: $F\colon W_n(A;p) \to W_{n-1}(A;p)$ is the unique ring map satisfying $w_k(Fx) = w_{k+1}(x)$. Equivalently, $F([a]) = [a^p]$ and F is multiplicative.

These satisfy the relations $RV = 0$, $FV = VF = p$, $\ker(R) = \mathrm{im}(V)$ and F is a lift of the Frobenius on A via the multiplicative lift map.

Definition 4.5.2. The *(infinite) p-typical Witt ring* is the limit $W(A;p) := \varprojlim_R W_n(A;p)$, with continuous maps R, F, V induced from the finite levels.

Example 4.5.3. For $\mathbb{F}_p$, $W_n(\mathbb{F}_p;p) \cong \mathbb{Z}/p^n$ and $W(\mathbb{F}_p;p) \cong \mathbb{Z}_p$ canonically as rings. Under these isomorphisms, R and F are reduction mod p^{n-1} and V is multiplication by p (followed by inclusion $\mathbb{Z}/p^{n-1} \to \mathbb{Z}/p^n$).

4.5.2 Witt Vectors as Fixed Points

We know from Theorem 4.2.15 that THH is cyclotomic. Taking $C_{p^{n-1}}$ fixed points of the isotropy separation sequence, as seen in §3.3, with A a commutative ring spectrum and $X = \mathrm{THH}(A)$, we get

$$\mathrm{THH}(A)_{hC_{p^n}} \to \mathrm{THH}(A)^{C_{p^n}} \xrightarrow{R} \mathrm{THH}(A)^{C_{p^{n-1}}}.$$

We show that for A connective, $\pi_0 \operatorname{THH}(A)_{hC_{p^n}} \simeq \pi_0 A$. For A connective, the cyclic bar construction gives that the 0-skeleton of $\operatorname{THH}(A)$ is just A. Then, each subsequent $A^{\wedge(s+1)}$ in the cyclic bar construction is connective, and attaching $s \geq 1$ simplices only adds suspensions $\Sigma^s A^{\wedge(s+1)}$, which are at least 1-connected. In particular, this means that these attachments cannot change π_0. We claim further that homotopy orbits do not change π_0 for connective spectra.

Lemma 4.5.4. *Let G be a compact Lie group and X a connective G–spectrum such that the G–action on $\pi_0 X$ is trivial. Then, $\pi_0 X_{hG} = \pi_0 X$.*

Proof. We filter EG by skeleta. The 0-skeleton is a free G-set G, so $EG^0_+ \wedge_G X \simeq G_+ \wedge_G X \simeq X$. Attaching higher s-cells adds wedges of $\Sigma^s X$, each of which is s-connected, so has $\pi_0 \Sigma^s X = 0$. Therefore the attachments cannot change π_0. Triviality of the G–action on $\pi_0 X$ ensures there is no further coinvariant quotient on π_0, so $\pi_0 X_{hG} \cong \pi_0 X$. □

For $X = \operatorname{THH}(A)$ with A connective, $\pi_0 \operatorname{THH}(A) \cong \pi_0 A$ and the S^1–action on π_0 is trivial, so $\pi_0 \operatorname{THH}(A)_{hC_{p^n}} \cong \pi_0 A$.

With this in mind, for A a commutative ring, $\operatorname{THH}(A)_{hC_{p^n}} \cong \pi_0 HA \cong A$.

Theorem 4.5.5 ([21, Thm. 2.3])**.** *Let A be a commutative ring. Then on π_0, this isotropy separation sequence*

$$\operatorname{THH}(A)_{hC_{p^n}} \to \operatorname{THH}(A)^{C_{p^n}} \xrightarrow{R} \operatorname{THH}(A)^{C_{p^{n-1}}}$$

becomes

$$0 \to A \xrightarrow{\iota} W_{n+1}(A;p) \xrightarrow{R} W_n(A;p) \to 0.$$

The inclusion ι identifies A with $\ker(R)$ *via $a \mapsto V^n([a])$.*

It turns out that this theorem can be generalised slightly. For A a connective, commutative ring spectrum, we get

$$0 \to \pi_0 A \xrightarrow{\iota} W_{n+1}(\pi_0 A;p) \xrightarrow{R} W_n(\pi_0 A;p) \to 0.$$

Using Theorem 4.5.5, we get the next theorem.

Theorem 4.5.6. *For $0 \leq k \leq n$, $\underline{\mathscr{A}}_{C_{p^n}}(C_{p^n}/C_{p^k}) \cong W_{k+1}(\mathbb{Z})$.*

Proof. Let $\mathbb{S}$ denote $\mathbb{S}$ with the trivial C_{p^n}-action. The unit map $\eta \colon \mathbb{S} \to H\mathbb{Z}$ induces a map of cyclotomic spectra $\operatorname{THH}(\mathbb{S}) \xrightarrow{\simeq} \mathbb{S} \to \operatorname{THH}(\mathbb{Z})$. Hence, for each $k \geq 0$ we get a natural ring map on fixed points

$$\alpha_k \colon \pi_0 \operatorname{THH}(\mathbb{S})^{C_{p^k}} \to \pi_0 \operatorname{THH}(\mathbb{Z})^{C_{p^k}}.$$

We know that $\pi_0 \operatorname{THH}(\mathbb{Z})^{C_{p^k}} \cong W_{k+1}(\mathbb{Z})$ and $\pi_0(\mathbb{S})^{C_{p^k}} \cong \underline{\mathscr{A}}_{C_{p^n}}(C_{p^n}/C_{p^k})$. It remains to be shown that α_k is an equivalence. It is clear that α_0 is an equivalence, so we proceed by induction. Consider the commutative diagram induced by η,

$$\begin{array}{ccccc} \operatorname{THH}(\mathbb{S})_{hC_{p^k}} & \longrightarrow & \operatorname{THH}(\mathbb{S})^{C_{p^k}} & \longrightarrow & \operatorname{THH}(\mathbb{S})^{C_{p^{k-1}}} \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{THH}(\mathbb{Z})_{hC_{p^k}} & \longrightarrow & \operatorname{THH}(\mathbb{Z})^{C_{p^k}} & \longrightarrow & \operatorname{THH}(\mathbb{Z})^{C_{p^{k-1}}}. \end{array}$$

We can then take π_* of the diagram, giving us two long exact sequences with induced maps between them. Recall that $\mathbb{S}$ and $H\mathbb{Z}$ are connective, so $\operatorname{THH}(\mathbb{S})$ and $\operatorname{THH}(\mathbb{Z})$ are connective. Also recall that $B \hookrightarrow W_k(B)$ is an injection. With this, we get the next diagram,

$$\begin{array}{ccccccccc} \pi_1 \operatorname{THH}(\mathbb{S})^{C_{p^{k-1}}} & \longrightarrow & \pi_0\mathbb{S} & \longrightarrow & \pi_0 \operatorname{THH}(\mathbb{S})^{C_{p^k}} & \longrightarrow & \pi_0 \operatorname{THH}(\mathbb{S})^{C_{p^{k-1}}} & \longrightarrow & 0 \\ \downarrow & & \downarrow{\cong} & & \downarrow{\alpha_k} & & \downarrow{\cong \text{ by induction}} & & \downarrow{\cong} \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & W_{k+1}(\mathbb{Z}) & \longrightarrow & W_k(\mathbb{Z}) & \longrightarrow & 0 \end{array}$$

By the five lemma, we find that α_k is an equivalence. Hence, $\underline{\mathscr{A}}_{C_{p^n}}(C_{p^n}/C_{p^k}) \cong W_{k+1}(\mathbb{Z})$. □

With knowledge of Witt vectors, we can extend parts of the computation from the last section 4.4.2 to any perfect field of characteristic p.

Theorem 4.5.7. *Let k be a perfect field of characteristic p. Then $\pi_0 \operatorname{TC}(k;p) \cong \mathbb{Z}_p$.*

Proof. Since k is p complete, we can work with the p complete definition of TC. Then, since $\operatorname{THH}(k)$ is cyclotomic, we have

$$\operatorname{TC}(k;p) \simeq \operatorname{hoeq}\left(\operatorname{THH}(k)^{hS^1} \underset{\varphi_p}{\overset{\text{can}}{\rightrightarrows}} \operatorname{THH}(k)^{tS^1} \right).$$

By Theorem 4.3.1, we know that $\pi_* \operatorname{THH}(k) \cong k[\mu]$ with $|\mu| = 2$, so odd groups vanish. Hence, on π_0 we get

$$\pi_0 \operatorname{TC}(k;p) \simeq \operatorname{eq}\left(W(k;p) \underset{F}{\overset{\text{id}}{\rightrightarrows}} W(k;p) \right) = W(k;p)^F$$

where F is the lift of the Frobenius $x \mapsto x^p$ from A to $W(A;p)$. At the level of $\mathbb{F}_p$ this looks like: $\mathbb{F}_p \to k \xrightarrow{\text{id}-F} k$, so at the level of Witt vectors this looks like: $\mathbb{Z}_p \to W(k;p) \xrightarrow{\text{id}-F} W(k;p)$. Since k is perfect, $F\colon k \xrightarrow{\cong} k$ is an isomorphism, and the fixed points $W(k;p)^F$ are canonically $W(\mathbb{F}_p;p) \cong \mathbb{Z}_p$. □

Chapter 5

Twisted Topological Hochschild Homology

This chapter develops a *twisted* form of topological Hochschild homology for C_n–ring spectra and organises the combinatorics of the twist using Λ_n–objects. Our presentation mainly follows [7], with minor variations to match the conventions used earlier in this thesis. The topological construction comes first: we introduce a twisted cyclic bar construction and a norm model. We then extract an algebraic analogue at the level of Green/Mackey functors and compare with classical Hochschild theory. Finally, we discuss the twisted cyclotomic structure.

5.1 Twisted THH

We introduce a twisted version of THH, for C_n-ring spectra.

Definition 5.1.1. Let R be a C_n-ring spectrum with multiplication map μ and unit map η. The *twisted cyclic bar construction* is a simplicial spectrum $B^{\mathrm{cy},C_n}_\bullet(R)\colon \Delta^{\mathrm{op}} \to \mathrm{Sp}$ given by $[q] \mapsto R^{\wedge(q+1)}$. Let $g \in C_n$ be the generator, and let α_q be the permutation of $R^{\wedge(q+1)}$ that cyclically permutes the last smash factor to the zeroth position and then acts by g on the new 0-th factor. The face and degeneracy maps are as follows:

$$d_i = \begin{cases} \mathrm{id}^{\wedge i} \wedge \mu \wedge \mathrm{id}^{\wedge(q-i-1)} & \text{for } 0 \le i < q, \\ (\mu \wedge \mathrm{id}^{\wedge(q-1)}) \circ \alpha_q & \text{for } i = q, \end{cases} \qquad s_i = \mathrm{id}^{\wedge(i+1)} \wedge \eta \wedge \mathrm{id}^{\wedge(q-i)}\,.$$

The *C_n-twisted topological Hochschild homology* of R, $\mathrm{THH}_{C_n}(R)$, is the geometric realisation of the twisted cyclic bar construction: $\mathrm{THH}_{C_n}(R) \coloneqq |B^{\mathrm{cy},C_n}_\bullet(R)|$.

In particular, this twisted cyclic bar construction is a variant of the cyclic bar construction that only differs in the last face map d_q.

We can think of this in terms of a norm construction too, where the model for the norm is exactly the twisted cyclic bar construction.

Definition 5.1.2 ([2, §8])**.** For R a C_n-ring spectrum, define the C_n-twisted THH of R as $\mathrm{THH}_{C_n}(R) = N_{C_n}^{S^1} R$.

Example 5.1.3. If R is a C_n-ring spectrum with trivial action, then the twisted cyclic bar construction reduces to the cyclic bar construction and $\mathrm{THH}_{C_n}(R) = \mathrm{THH}(R)$.

Computing nontrivial examples is outside the scope of this thesis. We refer to [1] for computational methods and further examples.

Unlike in the nonequivariant case where the algebraic Hochschild homology came before topological Hochschild homology, in this equivariant twisted case, the topological construction came first. We work backwards to try to discover what the algebraic analogue should be.

Definition 5.1.4 ([7, Def. 3.29])**.** Let $\underline{R}$ be a C_n-Green functor. The *C_n-twisted cyclic nerve* of $\underline{R}$, $\underline{\mathrm{HC}}_\bullet^{C_n}(\underline{R})$, is a simplicial C_n-Mackey functor with k-simplices $[k] \mapsto \underline{R}^{\Box(k+1)}$ given by the twisted cyclic bar construction with $\Box$ replacing $\wedge$.

We can extend this definition further to pairs of finite subgroups of S^1.

Definition 5.1.5 ([7, Def. 3.30])**.** Let $H \subset G \subset S^1$ be finite subgroups, and $\underline{R}$ an H-Green functor. The *G-twisted cyclic nerve* of $\underline{R}$ is

$$\underline{\mathrm{HC}}_H^G(\underline{R})_\bullet = \underline{\mathrm{HC}}_\bullet^G(N_H^G(\underline{R})).$$

The *G-twisted Hochschild homology* of $\underline{R}$ is

$$\underline{\mathrm{HH}}_H^G(\underline{R})_i = H_i(\underline{\mathrm{HC}}_H^G(\underline{R})_\bullet).$$

Conceptually, this models the restriction $R_G^{S^1}\,\mathrm{THH}_H(R)$ at the level of Green functors. In Definition 5.1.5, on the right hand side we are taking homology of a simplicial Mackey functor which is not *a priori* well-defined. However, we take the alternating sum of the face maps as our differential and homology of a chain complex of Mackey functors is taken for each subgroup. Equivalently, we can use the Dold–Kan correspondence to differentially graded Mackey functors and take homology of that [16, Cor. 2.3].

With this, we get an algebraic analogue of twisted THH. Now we can extend Theorem 4.2.16 to the equivariant setting:

Theorem 5.1.6 ([5, Thm. 1.1]). *Let $H \subset G \subset S^1$ be finite subgroups and R a connective, commutative H-ring spectrum. There is a natural map*

$$\underline{\pi}_k^G \operatorname{THH}_H(R) \to \underline{\operatorname{HH}}_H^G(\underline{\pi}_0^H(R))_k$$

which is an isomorphism for $k = 0$.

The authors then go on to extend the classic result of Hesselholt and Madsen Theorem 4.5.5 with [7, Prop. 6.5]. This proposition is outside the scope of this thesis, but the authors use [7, Lem. 6.10] to prove the proposition. We will focus on this lemma.

To simplify notation, for the remainder of this chapter, we write $N_k^n X$ and $i_k^n X$ in place of $N_{C_k}^{C_n} X$ and $i_{C_k}^{C_n} X$ respectively.

Lemma 5.1.7 ([7, Lem. 6.10]). *Let Q be a commutative $C_{p^{k-1}n}$-ring spectrum, where p is relatively prime to n. Let*

$$\eta_{p^{k-1}n}^{p^k n} \colon Q \to i_{p^{k-1}n}^{p^k n} N_{p^{k-1}n}^{p^k n} Q$$

be the unit of $C_{p^{k-1}n}$-commutative ring spectra adjunction and let

$$\varepsilon_{p^{k-1}}^{p^k} \colon N_{p^{k-1}}^{p^k} i_{p^{k-1}}^{p^k} R \to R$$

be the counit of C_{p^k}-commutative ring spectra adjunction. Then the restriction of the composite

$$N_{p^{k-1}}^{p^k} i_{p^{k-1}}^{p^{k-1}n} Q \xrightarrow[(with\ n)]{N_{p^{k-1}}^{p^k} i_{p^{k-1}}^{p^{k-1}n}\left(\eta_{p^{k-1}n}^{p^k n}\right)} N_{p^{k-1}}^{p^k} i_{p^{k-1}}^{p^k} i_{p^k}^{p^k n} N_{p^{k-1}n}^{p^k n} Q$$

$$\xrightarrow[(without\ n)]{\varepsilon_{p^{k-1}}^{p^k}} i_{p^k}^{p^k n} N_{p^{k-1}n}^{p^k n} Q$$

to $C_{p^{k-1}}$-ring spectra is

$$Q^{\wedge p} \xrightarrow{(g^{j_1} \wedge \ldots \wedge g^{j_p}) \circ \varphi} Q^{\wedge p}$$

where g is a generator of $C_{p^{k-1}n}$, φ is a permutation of the factors, and the j_i are integers.

In [7, Rmk. 6.11], the authors remark that this final formula relies on an explicit choice of the construction of the norm $N_{p^{k-1}n}^{p^k} Q$. This relies on an explicit choice of coset representatives to define the $C_{p^k n}$-action. With this in mind, let γ denote a generator of $C_{p^k n}$ and we note that the authors' choice of coset representatives was $\{1, \gamma, \gamma^2, ..., \gamma^{p-1}\}$. We explore consequences of their choice against our choice $\{1, \gamma^n, \gamma^{2n}, ..., \gamma^{(p-1)n}\}$. Let Q be a commutative $C_{p^{k-1}n}$-ring spectrum, then consider the below diagram:

$$\begin{array}{ccc}
N^{p^k}_{p^{k-1}} i^{p^{k-1}n}_{p^{k-1}} Q & \longrightarrow & N^{p^k}_{p^{k-1}} i^{p^k}_{p^{k-1}} \left(i^{p^k n}_{p^k} N^{p^k n}_{p^{k-1}n} Q \right) \\
\Big\downarrow \eta^{p^k n}_{p^{k-1}n} & \overset{=}{\nearrow} & \Big\downarrow \varepsilon_{p^k} \\
N^{p^k}_{p^{k-1}} i^{p^{k-1}n}_{p^{k-1}} \left(i^{p^k n}_{p^{k-1}n} N^{p^k n}_{p^{k-1}n} Q \right) & & i^{p^k n}_{p^k} N^{p^k n}_{p^{k-1}n} Q
\end{array}$$

Let ξ denote a generator of C_{p^k}, then in the norm $N^{p^k}_{p^{k-1}}$ we canonically choose $\{1, \xi, ..., \xi^{p-1}\}$ as coset representatives. Then mapping by the inclusion of $C_{p^k} \hookrightarrow C_{p^k n}$ we recover coset representatives $\{1, \gamma^n, ..., \gamma^{(p-1)n}\}$. Using this diagram, our choice of coset representatives appears to be the natural choice. With our choice, we recover a simpler formula:

Lemma 5.1.8. *Let Q be a commutative $C_{p^{k-1}n}$-ring spectrum, where p is relatively prime to n. Using coset representatives $\{1, \gamma^n, ..., \gamma^{(p-1)n}\}$, the restriction of the composite*

$$N^{p^k}_{p^{k-1}} i^{p^{k-1}n}_{p^{k-1}} Q \xrightarrow[(with\ n)]{N^{p^k}_{p^{k-1}} i^{p^{k-1}n}_{p^{k-1}} \left(\eta_{p^{k-1}n}\right)} N^{p^k}_{p^{k-1}} i^{p^k}_{p^{k-1}} i^{p^k n}_{p^k} N^{p^k n}_{p^{k-1}n} Q$$
$$\xrightarrow[(without\ n)]{\varepsilon^{p^k}_{p^{k-1}}} i^{p^k n}_{p^k} N^{p^k n}_{p^{k-1}n} Q$$

to $C_{p^{k-1}}$-ring spectra is

$$Q^{\wedge p} \xrightarrow{\mathrm{id}^{\wedge p}} Q^{\wedge p}.$$

Proof. Let $G = C_{p^k n} = \langle \gamma \rangle$, $H = C_{p^{k-1}n} = \langle \gamma^p \rangle$, and let Q be a commutative H-ring spectrum. We have $G/H \cong C_p = \langle \gamma \mid \gamma^p = e \rangle$, with coset representatives $\{e, \gamma^n, \gamma^{2n}, ..., \gamma^{(p-1)n}\}$. Let $h = \gamma^n$, the generator of $C_{p^k} \subset C_{p^k n}$. Now, we can consider the unit map $\eta^{p^k n}_{p^{k-1}n}$ as

$$Q \cong Q \wedge \mathbb{S}^{\wedge(p-1)} \xrightarrow{\mathrm{id} \wedge 1^{\wedge(p-1)}} Q^{\wedge p}.$$

Then, we can view $N^{p^k}_{p^{k-1}} i^{p^k n}_{p^{k-1}} N^{p^k n}_{p^{k-1}n} Q$ as p smash copies of $N^{p^k n}_{p^{k-1}n} Q$. We get

$$\begin{array}{ccc}
\left(N^{p^k n}_{p^{k-1}n} Q \right)^{\wedge p} & \xrightarrow{\varepsilon^{p^k}_{p^{k-1}}} & i^{p^k n}_{p^k} N^{p^k n}_{p^{k-1}n} Q \\
& \underset{\mathrm{id} \wedge h \wedge h^2 \wedge \ldots \wedge h^{p-1}}{\searrow} & \Big\uparrow \mathrm{mult} \\
& & \left(N^{p^k n}_{p^{k-1}n} Q \right)^{\wedge p}
\end{array}$$

where $h^i\colon N^{p^k n}_{p^{k-1}n}Q \to N^{p^k n}_{p^{k-1}n}Q$ is action by h^i. It remains to see how h^i acts with our choice of coset representatives. First we show how h acts, and to keep track of positions, we add a subscript:

$$h(e_1, \gamma_2^n, ..., \gamma_p^{(p-1)n}) = (\gamma^p e_p, \gamma^p \gamma_1^n, ..., \gamma^p \gamma_{p-1}^{(p-1)n})$$

So h is the permutation $(123\dots p)$. Therefore, the counit is given by the composite

$$(x_1, x_2, \dots, x_p) \mapsto \begin{pmatrix} x_1 & e & \cdots & e \\ x_2 & e & \cdots & e \\ \vdots & \vdots & \ddots & \vdots \\ x_p & e & \cdots & e \end{pmatrix} \xrightarrow{\mathrm{id}\wedge h \wedge h^2 \wedge \cdots \wedge h^{p-1}} \begin{pmatrix} x_1 & e & \cdots & e \\ e & x_2 & \cdots & e \\ \vdots & \vdots & \ddots & \vdots \\ e & e & \cdots & x_p \end{pmatrix}$$
$$\xrightarrow{\text{mult}} (x_1, x_2, \dots, x_p).$$

Hence, the composite is p-smash copies of the identity. □

This proof is slightly misleading since spectra do not have underlying sets. However, the permutations and actions can be best understood by looking at what the corresponding permutations and actions do to tuples.

We give an example of this theorem, and compare it with [7, Ex. 6.12].

Example 5.1.9. Let $p = 3$, $k = 1$, $n = 2$ and Q be a commutative C_2-ring spectrum. Then we have $G = C_6 = \langle\gamma\rangle$, $H = C_2 = \langle\gamma^3\rangle$ and $G/H \cong C_3$. We pick coset representatives $\{1, \gamma^2, \gamma^4\}$, so $h = \gamma^2$ generates $C_3 \subset C_6$. Now we can compute how γ and γ^2 act on the cosets:

$$\gamma\{1_1, \gamma_2^2, \gamma_3^4\} = \{\gamma^3 e_2, \gamma^3\gamma_3^2, \gamma^3\gamma_1^4\},$$
$$\gamma^2\{1_1, \gamma_2^2, \gamma_3^4\} = \gamma\{\gamma^3 e_2, \gamma^3\gamma_3^2, \gamma^3\gamma_1^4\} = \{\gamma^6 e_3, \gamma^6\gamma_1^2, \gamma^6\gamma_2^4\} = \{e_3, \gamma_1^2, \gamma_2^4\}.$$

With these coset representatives, we calculate $\varepsilon_e^3 \circ N_e^3 i_e^6(\eta_2^6)$ explicitly. Consider the unit map as

$$H\underline{R} \cong H\underline{R} \wedge \mathbb{S}^{\wedge 2} \xrightarrow{\mathrm{id}\wedge 1 \wedge 1} H\underline{R}^{\wedge 3},$$

so then $\varepsilon_e^3\colon N_e^3 i_e^6 N_2^6 H\underline{R} \to i_3^6 N_2^6 H\underline{R}$ is given by

$$\left(N_2^6 H\underline{R}\right)^{\wedge 3} \xrightarrow{\mathrm{id}\wedge h \wedge h^2} \left(N_2^6 H\underline{R}\right)^{\wedge 3} \xrightarrow{\text{mult}} i_3^6 N_2^6 H\underline{R}.$$

We can view $(N_2^6 H\underline{R})^{\wedge 3}$ as nine smash copies of $H\underline{R}$, so we think of this as

$$(x_1, x_2, x_3) \mapsto \begin{pmatrix} x_1 & e & e \\ x_2 & e & e \\ x_3 & e & e \end{pmatrix}.$$

Then we act by id, h, and h^2 on the rows in order. Since h is the permutation (123), h^2 is the permutation (132), so we get

$$\begin{pmatrix} x_1 & e & e \\ x_2 & e & e \\ x_3 & e & e \end{pmatrix} \xrightarrow{\mathrm{id} \wedge h \wedge h^2} \begin{pmatrix} x_1 & e & e \\ e & x_2 & e \\ e & e & x_3 \end{pmatrix} \xrightarrow{\text{mult}} (x_1, x_2, x_3).$$

Hence the map is given by

$$\varepsilon_e^3 \circ N_e^3 i_e^6(\eta_2^6) \colon H\underline{R}^{\wedge 3} \xrightarrow{\mathrm{id}^{\wedge 3}} H\underline{R}^{\wedge 3}.$$

This is notably simpler than the formula in Example [7, Ex. 6.12] which is

$$H\underline{R}^{\wedge 3} \xrightarrow{(\mathrm{id} \wedge g \wedge \mathrm{id}) \circ \varphi} H\underline{R}^{\wedge 3}$$

where φ is the permutation (1)(23).

We note that with coset representatives $\{e, \gamma, \gamma^2, \ldots, \gamma^{p-1}\}$, γ acts on the cosets just by a cyclic permutation and acting by the generator on any that cycled around. This exactly matches the twisted cyclic bar construction, which was convenient for the authors' proof of [7, Prop. 6.5].

However, with the simplified formula in mind, we propose our coset representatives $\{e, \gamma^n, \ldots, \gamma^{(p-1)n}\}$ as the "categorical" choice. The following is a well known result in category theory:

Theorem 5.1.10. *Let L and R be left and right adjoint functors between categories $\mathcal{C}$ and $\mathcal{D}$*

$$L \colon \mathcal{C} \rightleftarrows \mathcal{D} \colon R$$

then the composite

$$LX \xrightarrow{L\eta_X} LRLX \xrightarrow{\varepsilon_{LX}} LX$$

is the identity.

Proof. Let $A \in \mathcal{C}$ and $B \in \mathcal{D}$. Write the adjunction as

$$\Phi_{A,B} \colon \mathrm{Hom}_{\mathcal{D}}(LA, B) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{C}}(A, RB).$$

By definition, $\eta_A = \Phi_{A,LA}(\mathrm{id}_{LA})$, $\varepsilon_B = \Phi^{-1}_{RB,B}(\mathrm{id}_{RB})$. Fix $X \in \mathcal{C}$ and let $\alpha :=$ $\varepsilon_{LX} \circ L\eta_X \colon LX \to LX$. Apply $\Phi_{X,LX}$ to α, $\Phi_{X,LX}(\alpha) = R(\varepsilon_{LX}) \circ \eta_X$. Naturality of η gives $R(\varepsilon_{LX}) \circ \eta_X = \eta_X$ because $R(\varepsilon_{LX}) \colon RLRLX \to RLX$ folds the extra RL factor introduced by η_X. Thus, $\Phi_{X,LX}(\alpha) = \eta_X = \Phi_{X,LX}(\mathrm{id}_{LX})$. Since $\Phi_{X,LX}$ is a bijection, it follows that $\alpha = \mathrm{id}_{LX}$. Hence $\varepsilon_{LX} \circ L\eta_X = \mathrm{id}_{LX}$, which is the required triangle identity. ∎

Theorem 5.1.10 shows that categorically, we expect the identity in a composition like this.

In the authors' proof of [7, Prop. 6.5], they implicitly use a special case of the next theorem without reference or proof.

Theorem 5.1.11. *Let $G = C_{mn}$ and $H = C_n \subset G$. Let R be a commutative H-ring spectrum. There is a natural isomorphism of simplicial H-spectra*

$$i_H^G B_\bullet^{\mathrm{cy},G}(N_H^G R) \cong sd_m B_\bullet^{\mathrm{cy},H}(R)$$

and hence

$$i_H^G \,\mathrm{THH}_G(N_H^G R) \simeq sd_m \,\mathrm{THH}_H(R) \simeq \mathrm{THH}_H(R).$$

The idea behind Theorem 5.1.11 matches up with the norm definition of HH_H^G from Definition 5.1.5.

Proof. Let $G = C_{mn} = \langle \gamma \rangle$ then $H = \langle \gamma^m \rangle$ and the quotient $G/H \cong C_m = \langle \gamma \mid \gamma^m \rangle$. Let R be a commutative H-ring spectrum. To prove this theorem, we will give a map $f_\bullet \colon i_H^G B_\bullet^{\mathrm{cy},G}(N_H^G R) \to sd_m B_\bullet^{\mathrm{cy},H}(R)$ that is an isomorphism of H-spectra and is simplicial; that is, satisfying the simplicial identities.

With the coset representatives $\{e, \gamma, ..., \gamma^{m-1}\}$ for G/H, we use the indexed smash for the norm

$$N_H^G R \simeq \bigwedge_{j=0}^{m-1} R_{(j)}$$

where each $R_{(j)}$ is a copy of R as an H-spectrum and γ permutes the factors cyclically, $\gamma R_{(j)} = R_{(j+1)}$.

Let

$$X_q \coloneqq i_H^G B_q^{\mathrm{cy},G}(N_H^G R) \simeq \bigwedge_{k=0}^{q} \bigwedge_{j=0}^{m-1} R_{(k,j)}$$

where k indexes the $q+1$ bar-factors and j indexes the coset coordinates inside the norm. Let

$$Y_q \coloneqq (sd_m B_\bullet^{\mathrm{cy},H}(R))_q = B_{m(q+1)-1}^{\mathrm{cy},H}(R) \simeq \bigwedge_{l=0}^{m(q+1)-1} R_{(l)}.$$

Arrange the $m(q+1)$ factors as a rectangle with $q+1$ rows (indexed by k) and m columns (indexed by j),

$$\{(k,j) \mid 0 \le k \le q, 0 \le j \le m-1\}.$$

The X_q side reads the smash in row-major order (group first by k) while Y_q with edgewise subdivision reads in column-major order (group first by j). Define $f_q\colon X_q \to Y_q$ by $R_{k,j} \mapsto R_{j(q+1)+k}$. Equivalently, we can write this as a permutation of $\{0, ..., m(q+1)-1\}$, with $\sigma_{q,m}\colon km+j \mapsto j(q+1)+k$. It remains to show that f_q agrees with the simplicial identities in order to assemble into a simplicial isomorphism.

In X_q, for $i < q$ the d_i multiplies the i-th and $(i+1)$-st smash factors, that is, for each fixed column j, it multiplies $R_{(i,j)}$ with $R_{(i+1,j)}$. For $i = q$ it cycles the last smash factor to the front, gets acted on by h and multiplies with the first smash factor. The face maps are given by

$$d_i^X(R_{(0,j)}, ..., R_{(q-2,j)}, R_{(q-1,j)}) = \begin{cases} (..., R_{(i,j)}R_{(i+1,j)}, ...) & 0 \leq i < q, \\ (hR_{(q-1,j)}R_{(0,j)}, ..., R_{(q-2,j)}) & i = q. \end{cases}$$

Under f_q, these two entries go to positions $j(q+1)+i$ and $j(q+1)+i+1$ in Y_q. By definition of edgewise subdivision

$$d_i^{sd} = d_i d_{i+(q+1)} ... d_{i+(m-1)(q+1)}\colon Y_q \to Y_{q-1}$$

which (reading by columns) exactly multiplies the pairs of indices $j(q+1)+i$ and $j(q+1)+i+1$ in each column. Hence, $f_{q-1}d_i = d_i^{sd} f_q$.

In X_q, d_q uses the cyclic operator to bring the last row $k = q$ to the front and then multiplies with row $k = 0$ and gets acted on by h. Columnwise, this multiplies $hR_{(q,j)}$ with $R_{(0,j)}$ for each j. Under f_q, these land at indices $j(q+1)+q$ and $j(q+1)+0$, the last and first entries within the j-th column block of length $q+1$. On the Y side

$$d_q^{sd} = d_q d_{q+(q+1)} ... d_{q+(m-1)(q+1)}.$$

For each $j < m-1$, $d_{q+j(q+1)}$ multiplies the adjacent pair inside that column block, and for $j = m-1$, the index is $q+(m-1)(q+1) = m(q+1)-1$ which is the last face on $B^{\mathrm{cy},H}_{m(q+1)-1}(R)$ and therefore multiplies the last and the first entries and acts by h, exactly the required wrap around. Thus, $f_{q-1}d_q = d_q^{sd} f_q$.

The degeneracy maps are given by inserting unit maps in each column. The same columnwise reindexing shows $f_{q+1}s_i = s_i^{sd} f_q$.

Thus, $f_\bullet$ assemble to a simplicial isomorphism. After geometric realisation we get the equivalence $i_H^G \operatorname{THH}_G(N_H^G R) \simeq sd_m \operatorname{THH}_H(R)$. Finally, using Theorem 4.1.13, we get $sd_m \operatorname{THH}_H(R) \simeq \operatorname{THH}_H(R)$. □

To illustrate what is happening, we give an example of this theorem.

Example 5.1.12. Let $m = 3$, $n = 2$ and R be a commutative ring spectrum. So $G = C_6 = \langle\gamma\rangle$, $H = C_2 = \langle\gamma^3\rangle$ and $G/H \cong C_3 = \langle\gamma \mid \gamma^3\rangle$. We consider the step with $q = 1$. Then $X_1 = i_2^6 B_q^{\mathrm{cy},C_6}(N_2^6 R) \simeq (R^{\wedge 3})^{\wedge 2}$, written as two blocks of three, with d_1 given by

$$\begin{pmatrix} r_1^0 & r_1^1 \\ r_2^0 & r_2^1 \\ r_3^0 & r_3^1 \end{pmatrix} \mapsto \begin{pmatrix} (\gamma^3 r_3^1) r_1^0 \\ r_1^1 r_2^0 \\ r_2^1 r_3^0 \end{pmatrix}.$$

On the subdivision side $Y_1 = (sd_3 B_\bullet^{\mathrm{cy},C_2}(R))_1 = B_5^{\mathrm{cy},C_2}(R) \simeq R^{\wedge 6}$ and $d_1^{sd} = d_1 \circ d_3 \circ d_5$, given by $\begin{pmatrix} r_0 & r_1 & r_2 & r_3 & r_4 & r_5 \end{pmatrix} \mapsto \begin{pmatrix} \gamma^3 r_5 r_0 & r_1 r_2 & r_3 r_4 \end{pmatrix}$. We can package this as a matrix so that rows come from subdivision and columns from simplicial degree:

$$\begin{pmatrix} r_0 & r_1 \\ r_2 & r_3 \\ r_4 & r_5 \end{pmatrix} \mapsto \begin{pmatrix} \gamma^3 r_5 r_0 \\ r_1 r_2 \\ r_3 r_4 \end{pmatrix}.$$

With the identifications given in the proof, these match up.

Remark 5.1.13. If we take $G = C_{p^k n}$ and $H = C_{p^{k-1}n} \subset G$, then the coset representatives used in the proof of Theorem 5.1.11 are $\{e, \gamma, ..., \gamma^{p-1}\}$, the same as those in the paper [7]. In particular, this provides another good reason to use their choice of coset representatives.

Remark 5.1.14. Let $G = C_{p^k}$, $H = e$ and consider $H\mathbb{F}_p$. We find that $i_e^{p^k} \mathrm{THH}_{C_{p^k}}(N_e^{p^k} H\mathbb{F}_p) \simeq \mathrm{THH}(\mathbb{F}_p)$. We computed the right hand side in Chapter 4, and we saw that taking fixed points gives the Witt vectors. Theorem 5.1.11 may give a way to recover this Witt vector result through purely equivariant means, though we leave this to future work.

5.2 **Twisted** TC

In the untwisted case, TC is obtained from THH by using its cyclotomic structure. The same strategy is not so clear in the twisted setting, because the cyclotomic Frobenius must account for the p-th power automorphism $\phi_p \colon C_n \to C_n$. In the twisted case, we show that THH is twisted p-cyclotomic.

As explained in [7, Thm. 3.4], there is a subtlety in how the cyclotomic structure is produced in the C_n-twisted setting: one must account for the twist by the p-th power automorphism on C_n. The original paper on this subject missed the twist [2, Thm. 8.6]. The correction to the p-cyclotomic structure map is given precisely below:

Theorem 5.2.1 ([7, Thm. 3.4]). *Let R be a C_n-ring spectrum and let p be prime with $p \nmid n$. Then there is a natural map*

$$\mathrm{THH}_{C_n}(R) \longrightarrow \rho_p^* \Phi^{C_p} \, \mathrm{THH}_{C_n}\left(\phi_p^* R\right)$$

that is an equivalence of $C_{p^\infty n}$-spectra. Here:

- *$\phi_p \colon C_n \to C_n$ is the p-th power map and ϕ_p^* means "regard R as a C_n-spectrum via ϕ_p": the C_n-action on R is precomposed with ϕ_p.*
- *ρ_p^* is inflation/restriction of groups: we regard a $C_{p^\infty n}/C_p$-spectrum as a $C_{p^\infty n}$-spectrum along the quotient $C_{p^\infty n} \to C_{p^\infty n}/C_p$.*

However, this cyclotomic map is off by a twist. To fix this, we iterate the maps and since $p \nmid n$, the endomorphism ϕ_p of C_n is an automorphism of finite order ν, given by the multiplicative order of p modulo n. Then Theorem 5.2.1 implies that iterating the previous construction gives a $C_{p^\infty n}$-equivalence

$$\mathrm{THH}_{C_n}(R) \xrightarrow{\simeq} \rho_{p^\nu}^* \Phi^{C_{p^\nu}} \, \mathrm{THH}_{C_n}(R).$$

More generally, this motivates the definition of (genuine) twisted p-cyclotomic spectra:

Definition 5.2.2. For a fixed prime p and integer n with $p \nmid n$, a *(genuine) C_n-twisted p-cyclotomic spectrum* is a genuine S^1-spectrum X with an S^1-map $X \to \rho_p^* \Phi^{C_p}(\phi_p^* X)$.

In Chapter 4 we used cyclotomic structure to define TC, so it is natural to try to use twisted p-cyclotomic structure to define twisted TC_{C_n}. In [7, Rmk. 3.6], the authors describe the *restriction* maps $R^\nu \colon \mathrm{THH}_{C_n}(R)^{C_{p^k}} \to \mathrm{THH}_{C_n}(R)^{C_{p^{k-\nu}}}$. Though it is unclear how to define twisted TC_{C_n} as a limit over R and F now, since we need to take the limit over every ν-th term. Consider the case of $\nu = 2$, then R^2, F^2 and $RF = FR$ are all maps that we should consider in the limit; however, RF still has a twist. With this in mind, we propose the following definition of twisted TC:

Definition 5.2.3. Let X be a C_n-twisted p-cyclotomic spectrum. Then the *C_n-twisted topological cyclic homology of X at p* is

$$\mathrm{TC}_{C_n}(X;p) := \underset{R^\nu, F^\nu}{\underleftarrow{\mathrm{holim}}} \left(X^{C_{p^{k\nu}}} \to X^{C_{p^{(k-1)\nu}}} \right),$$

with Frobenius $F^\nu\colon X^{C_{p^{k\nu}}} \to X^{C_{p^{(k-1)\nu}}}$ corresponding to inclusion of fixed points and restriction $R^\nu\colon X^{C_{p^{k\nu}}} \to X^{C_{p^{(k-1)\nu}}}$ arising from the composition $X^{C_{p^{k\nu}}} \to \left(\Phi^{C_{p^\nu}} X\right)^{C_{p^{(k-1)\nu}}} \xrightarrow{\simeq} X^{C_{p^{(k-1)\nu}}}$.

To match previous TC discussions, we want to bring this into the Nikolaus–Scholze framework and use a homotopy equaliser definition as in Definition 4.4.2. In this framework, we expect to need to find a $C_{p^\infty n}$-map $\varphi_p\colon \mathrm{THH}_{C_n}(R) \to \rho_p^* \mathrm{THH}_{C_n}(R)^{tC_p}$. From §3.3 there is always a natural map $\Phi^{C_p}\mathrm{THH}_{C_n}(R) \to \mathrm{THH}_{C_n}(R)^{tC_p}$. Composing this map with the equivalence of Theorem 5.2.1 yields the desired cyclotomic Frobenius

$$\varphi_p\colon \mathrm{THH}_{C_n}(R) \xrightarrow{\simeq} \rho_p^*\Phi^{C_p}\ \mathrm{THH}_{C_n}\left(\phi_p^* R\right) \to \mathrm{THH}_{C_n}(\phi_p^* R)^{tC_p},$$

which is $C_{p^\infty n}$-equivariant by construction.

However, again, the Frobenius is off by a twist. In trying to fit this into the theory, we step back to the nontwisted case. Let T be a p-cyclotomic spectrum. Then by [34, Cor. II.4.7], the following diagram encodes the structure that produces TC:

$$\begin{array}{ccccccc}
T^{C_{p^k}} & & & \xrightarrow{\quad R^k \quad} & & & \Phi^{C_{p^k}}T \\
\downarrow & & & & & & \downarrow \\
 & & & & (\Phi^{C_{p^{k-1}}}T)^{hC_p} & \to & (\Phi^{C_{p^{k-1}}}T)^{tC_p} \\
 & & & & \downarrow & & \\
 & & (\Phi^{C_{p^2}}T)^{hC_{p^{k-2}}} & \longrightarrow & \cdots & & \\
 & & \downarrow & & & & \\
 & (\Phi^{C_p}T)^{hC_{p^{k-1}}} \to & \left((\Phi^{C_p}T)^{tC_p}\right)^{hC_{p^{k-2}}} & & & & \\
 & \downarrow & & & & & \\
T^{hC_{p^k}} \to & (T^{tC_p})^{hC_{p^{k-1}}} & & & & &
\end{array}$$

In our twisted setting, the only extra bookkeeping is the appearance of the twists ϕ_p^*. We see that the Nikolaus–Scholze definition of a p-cyclotomic spectrum holds information about $T^{C_{p^k}}$ and restriction maps R^k for all k [34, Cor. II.4.7]. On the other hand, in the twisted setting we need to twist ν times to get back an untwisted version. This means that we can only draw the corresponding diagram when $\nu \mid k$. Hence, this version of twisted p-cyclotomic only holds information about $T^{C_{p^{k\nu}}}$ and $R^{k\nu}$.

With this, we give a Nikolaus–Scholze style definition of twisted p-cyclotomic.

Definition 5.2.4. For a fixed prime p and integer n with $p \nmid n$, a *C_n-twisted p-cyclotomic spectrum* is a genuine $C_{p^\infty n}$-spectrum X with a $C_{p^\infty n}$-map $\varphi_p \colon X \to \rho_p^*(\phi_p^* X)^{tC_p}$.

The difficulty now lies in defining twisted TC_{C_n} in terms of a homotopy equaliser. As in the nontwisted case, the canonical map can: $X^{hC_{p^\infty}} \to X^{tC_{p^\infty}}$ is inclusion of homotopy fixed points into Tate. However, the Frobenius is $\varphi_p \colon X \to \phi_p^* X^{tC_p}$. One could try to rectify this by only looking at the ν-th power of p, but this does not remove the twist:

$$\begin{aligned}
\varphi_{p^\nu} : X &\xrightarrow{\simeq} \Phi^{C_{p^\nu}} X \\
&= \Phi^{C_p} \Phi^{C_{p^{\nu-1}}} X \\
&\to \left(\Phi^{C_{p^{\nu-1}}} X\right)^{tC_p} \\
&\simeq \left(\left(\phi^*_{p^{\nu-1}}\right)^{-1} X\right)^{tC_p} \\
&= (\phi_p^* X)^{tC_p}
\end{aligned}$$

Instead, we provide the below theorem detailing how this works:

Theorem 5.2.5. *Let X be a C_n-twisted p-cyclotomic spectrum, then the C_n-twisted topological cyclic homology at p is*

$$\mathrm{TC}_{C_n}(X;p) \simeq \operatorname{hoeq}\left(\bigvee_{i=0}^{\nu-1} \left(\phi^*_{p^i} X\right)^{hC_{p^\infty}} \underset{(\varphi_p)^{hC_{p^\infty}}}{\overset{\mathrm{can}}{\rightrightarrows}} \bigvee_{i=0}^{\nu-1} \left(\phi^*_{p^i} X^{tC_p}\right)^{hC_{p^\infty}} \right)$$

with standard canonical map $\phi^*_{p^i} X^{hC_{p^\infty}} \to \phi^*_{p^i}(X^{tC_p})^{hC_{p^\infty}}$, and Frobenius map $\phi^*_{p^i} X \to \phi^*_{p^{i+1}} X^{tC_p}$ with indices of p taken modulo ν, where $\phi^*_{p^0} X = X$.

We can think of the homotopy equaliser as

$$\operatorname{hoeq}\left(\begin{matrix} X^{hC_{p^\infty}} \\ \vee \\ \phi_p^* X^{hC_{p^\infty}} \\ \vee \\ \vdots \\ \vee \\ \phi^*_{p^{\nu-1}} X^{hC_{p^\infty}} \end{matrix} \quad \underset{(\varphi_p)^{hC_{p^\infty}}}{\overset{\mathrm{can}}{\rightrightarrows}} \quad \begin{matrix} \left(X^{tC_p}\right)^{hC_{p^\infty}} \\ \vee \\ \left(\phi_p^* X^{tC_p}\right)^{hC_{p^\infty}} \\ \vee \\ \vdots \\ \vee \\ \left(\phi^*_{p^{\nu-1}} X^{tC_p}\right)^{hC_{p^\infty}} \end{matrix} \right)$$

with the canonical map going directly across levelwise, and the Frobenius map going across levelwise and down a twisted level with the last factor cycling to the top.

Proof. We adapt the proof of [34, Thm II.4.10]. We first consider the square:

$$\begin{array}{ccc}
\bigvee_{i=0}^{k\nu} \phi_{p^i}^* X^{hC_{p^{k\nu-i}}} & \xrightarrow{R'-F'} & \bigvee_{i=0}^{(k-1)\nu} \phi_{p^i}^* X^{hC_{p^{k\nu-i}}} \\
\downarrow{\scriptstyle \varphi_p - \mathrm{can}} & & \downarrow{\scriptstyle \varphi_p - \mathrm{can}} \\
\bigvee_{i=1}^{k\nu} \left(\phi_{p^i}^* X^{tC_p}\right)^{hC_{p^{k\nu-i}}} & \xrightarrow{R''-F''} & \bigvee_{i=1}^{(k-1)\nu} \left(\phi_{p^i}^* X^{tC_p}\right)^{hC_{p^{k\nu-i}}}
\end{array}$$

Here, R' forgets the first ν factors, and F' forgets the last ν factors and projects $X^{hC_{p^{j\nu}}} \to X^{hC_{p^{(j-1)\nu}}}$. R'' forgets the first ν factors and F'' forgets the last factor and projects $(X^{tC_p})^{hC_{p^{j\nu}}} \to (X^{tC_p})^{hC_{p^{(j-1)\nu}}}$. We see commutativity of this diagram by checking every factor separately. Then computing the horizontal and vertical fibers yields the below diagram.

$$\begin{array}{ccccc}
\mathrm{hofib} & \longrightarrow & X^{C_{p^{k\nu}}} & \xrightarrow{R^\nu - F^\nu} & X^{C_{p^{(k-1)\nu}}} \\
\downarrow & & \downarrow & & \downarrow \\
\bigvee_{i=0}^{\nu-1} \phi_{p^i}^* X^{hC_{p^{k\nu-i}}} & \longrightarrow & \bigvee_{i=0}^{k\nu} \phi_{p^i}^* X^{hC_{p^{k\nu-i}}} & \xrightarrow{R'-F'} & \bigvee_{i=\nu}^{k\nu} \phi_{p^i}^* X^{hC_{p^{k\nu-i}}} \\
\downarrow{\scriptstyle \varphi_p - \mathrm{can}} & & \downarrow{\scriptstyle \varphi_p - \mathrm{can}} & & \downarrow{\scriptstyle \varphi_p - \mathrm{can}} \\
\bigvee_{i=1}^{\nu} \left(\phi_{p^i}^* X^{tC_p}\right)^{hC_{p^{k\nu-i}}} & \longrightarrow & \bigvee_{i=1}^{k\nu} \left(\phi_{p^i}^* X^{tC_p}\right)^{hC_{p^{k\nu-i}}} & \xrightarrow{R''-F''} & \bigvee_{i=\nu+1}^{k\nu} \left(\phi_{p^i}^* X^{tC_p}\right)^{hC_{p^{k\nu-i}}}
\end{array}$$

By design, the map on the vertical homotopy fibers is $R^\nu - F^\nu$, the maps from Definition 5.2.3. Then, by commutativity of the diagram we get,

$$\mathrm{hofib} \simeq \mathrm{hoeq}\left(\bigvee_{i=0}^{\nu-1} \phi_{p^i}^* (X)^{hC_{p^{k\nu-i}}} \underset{(\varphi_p)^{hC_{p^{k\nu}}}}{\overset{\mathrm{can}}{\rightrightarrows}} \bigvee_{i=1}^{\nu} \phi_{p^i}^* \left(X^{tC_p}\right)^{hC_{p^{k\nu-i}}} \right).$$

In the limit this yields,

$$\begin{aligned}
\mathrm{hofib} &\simeq \mathrm{TC}_{C_n}(X;p) \\
&\simeq \mathrm{hoeq}\left(\bigvee_{i=0}^{\nu-1} \phi_{p^i}^* (X)^{hC_{p^\infty}} \underset{(\varphi_p)^{hC_{p^\infty}}}{\overset{\mathrm{can}}{\rightrightarrows}} \bigvee_{i=1}^{\nu} \phi_{p^i}^* \left(X^{tC_p}\right)^{hC_{p^\infty}} \right)
\end{aligned}$$

as required. □

Example 5.2.6. For $n = 1$ and X a bounded-below C_n-twisted p-cyclotomic spectrum, we have $\nu = 1$ and we recover the untwisted $\mathrm{TC}(X;p)$.

5.3 Future Directions

The original twisted p-cyclotomic framework, original definition and original description of twisted TC_{C_n}, suggest several lines for future directions. Most importantly, determining a twisted cyclotomic trace map from $K_{C_n} \to \mathrm{TC}_{C_n}$, restricting to the ordinary cyclotomic trace. Computing basic, nontrivial examples of twisted TC_{C_n} and comparing these computations to the nontwisted case. With an appropriate notion of twisted Witt vectors, determining whether $\pi_0 \mathrm{TC}_{C_n} \cong W^{(\phi)}$, analogous to the nontwisted case.

As mentioned in Remark 5.1.14, there is possibility of applying our Theorem 5.1.11 to recover the result of Hesselholt and Madsen, through purely equivariant means.

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