LÉVY INTEGRATION

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1. Introduction

Dear Pierre,

For our report we have decided on the topic of Lévy integrals - specifically their construction and applications. The topic intrigued us as the Lévy integral is a powerful tool and a natural extension of the Itô integral, yet the number of references are sparse and give little intuition to the relevant components. We hope to give a streamlined overview of the Lévy integral in one dimension, and provide intuition along the way.

We will begin by introducing the new concept of Poisson integration as well as several related constructions needed to define the Lévy integral. Then we review the Lévy-Itô decomposition and further extend our understanding by showing that, aside from the Brownian, it decomposes further into a compound Poisson process and a compensated Poisson process. Finally, we conclude with the construction of Lévy-type stochastic integrals, and an application of it to options pricing.

2. Poisson integration

As a spoiler for what is coming up in section 3, the Lévy-Itô decomposition gives us that all Lévy processes are the sum of a Brownian motion with drift, a compound Poisson process and a compensated Poisson process. In this section, we construct the Poisson integral, which (loosely speaking) aggregates the contribution of jumps which occur at random times in the Lévy process.

First - recall the Compound Poisson process. This process is vital in constructing the Lévy integral as this type of process contributes to all the jumps of a Lévy process.

Definition

A **compound Poisson process** is defined as $Y(t) = \sum_{i=1}^{N(t)} X_i$, where $N(t)$ is a Poisson process and X_i are i.i.d. random variables independent to $N(T)$. The **intensity** λ of $Y(t)$ is the intensity of $N(t)$.

2.1. **Poisson measure.** In order to construct the Poisson integral, we introduce the concept of a Poisson measure. This is an example of a random measure - a function which assigns to each set of an sigma algebra a random variable. However, when we fix an $\omega \in \Omega$, the random measure turns into a measure on the underlying space.

Definition

Let (E, σ, μ) be a measure space with $\sigma-$ finite measure μ. Then a family of random variables $\{N_A\}_{A\in\Sigma}$ on some probability space Ω is called the Poisson random measure with intensity μ if:

(1) $\forall A \in \Sigma, N_A$ is a Poisson variable with mean λ
(2) If A_1, A_2 are disjoint, then N_{A_1} is independent to N_{A_w}.
(3) $\forall \mu \in \Omega, N_-(\omega)$ is a measure on (E, σ).

A crude idea for defining the Lévy integral is to integrate against the Brownian and the Jump process separately. Since a process like a compound Poisson process only has jumps at discrete times, defining a Poisson integral is in some ways easier than defining the Itô integral. However, we need a way of keeping track of when jumps occurs. To do so, we associate to each Lévy process $L(t)$ a process ΔL which keeps tracks of all the jumps which occur in its paths:

$$\Delta L(t) = L(t) - \lim_{x\to t^-} L(x)$$

We will write $L(t-)$ to denote $\lim_{x\to t^-} L(x)$ (since it comes up a lot). Note that ΔL is non zero precisely when the path is discontinuous, which is when a jump occurs.

Then doing this, we can associate a Poisson measure N to L:

$$N(t, A)(\omega) := \sum_{0\leq s\leq t} \chi_A(\Delta X(s)(\omega))$$

Where A is a measurable set in $\mathbb{R} \setminus \{0\}$ such that $0 \notin \bar{A}$. We call such an A **bounded below**. We will not prove that this indeed defines a Poisson measure as it is far too long. One can see that this measure encodes the number of times the path of a Lévy process jumps with magnitude in A before time t.

Prop. 1. *$N(t, A)$ defines a Poisson process.*

See that $\mu := \mathbb{E}(N(t, \cdot))$ is a Borel measure on $\mathbb{R}\setminus\{0\}$ which we will call the **intensity measure** associated to L. Then **compensated Poisson random measure** associated to L is then defined as:

$$\tilde{N}(t, A) = N(t, A) - t\mu(A)$$

By removing drift term from $N(t, A)$, we see that the process defined by a compensated Poisson measure has zero mean and in fact $\tilde{N}(t, A)$ is a Martingale called a compensated Poisson process. Such a process measures the deviation of the underlying Poisson process from its mean.

2.2. **Integration.** Now we define the Poisson integral with respect to a deterministic function.

Definition

Let $f : \mathbb{R} \to \mathbb{R}$ be measurable and let A be bounded below. Let N be the Poisson measurable associated to a Lévy process. Then we define the **Poisson integral** of f as:

$$\int_A f(x)N(t,dx)(\omega) := \sum_{x \in A} f(x)N(t,\{x\})(\omega)$$

Definition

Let the setup be as above. We define the **compensated Poisson integral** to be

$$\int_A f(x)\tilde{N}(t,dx)(\omega) := \int_A f(x)\tilde{N}(t,dx)(\omega) - t\int_A f(x)\mu(dx)$$

Then extending from our interpretation of the Poisson measure N, we see that the Poisson intergal of f is simply the aggregate sum of the jumps weighted by f, under the restriction that the jumps have value in A.

Theorem

Let $f_1 \in L^1(A, \mu|_A)$, $f_2 \in L^1(B, \mu|_B)$. Define $\mu_{f,A}(C) = \mu(A \cap f^{-1}(C))$. Then the characteristic functions of $Y(t) := \int_A f_1(x)N(t,dx)$ and $Z(t) := \int_B f_2(x)\tilde{N}(t,dx)$ are given by:

$$\phi_{Y(t)} = \exp\left(t\int_{\mathbb{R}}(e^{i\theta x} - 1)\mu_{f_1,A}(dx)\right)$$

$$\phi_{Z(t)} = \exp\left(t\int_{\mathbb{R}}(e^{i\theta x} - 1 - i\theta x)\mu_{f_2,B}(dx)\right)$$

Proof. Let $f = \sum_{i=1}^n c_i\chi_{A_i}$ be a simple function and WLOG assume that the A_i are disjoint. Then we have:

$$\mathbb{E}\left(\exp\left[i\theta\int_A f_1(x)N(t,dx)\right]\right) = \mathbb{E}\left(\exp\left[i\theta\int_A(x)\sum_{i=1}^n c_iN(t,A_j)\right]\right) \tag{1}$$

$$= \prod_{i=1}^n \mathbb{E}\left(\exp\left[i\theta c_iN(t,A_j)\right]\right) \tag{2}$$

$$= \prod_{i=1}^n \exp\left(t\left(\exp(i\theta c_i) - 1\right)\mu(A_j)\right) \tag{3}$$

$$= \exp\left(t\int_A(\exp(i\theta f_1(x)) - 1)\mu(dx)\right) \tag{4}$$

$$= \exp\left(t\int_{\mathbb{R}}(\exp(i\theta x - 1)\mu_{f_1,A}(dx)\right) \tag{5}$$

The third equality comes from recognising that it is simply the characteristic function of a Poisson random variable with $\lambda = ln(c_j)\mu(A_j)$. Then for any arbitrary $f_1 \in L^1(A, \mu_A)$ can approximate f_1 with simple functions, then taking the limit and applying dominated convergence theorem yields the first result.

Differentiating the first result with f_2 and B instead of f_1 and A gives us

$$\mathbb{E}\left(\int_B f_2(x)N(t,dx)\right) = t\int_B f_2(x)\mu dx$$

and the second result follows immediately. □

Remark. *This theorem tells us that the* $\int_A f(x)N(t,dx)$ *has the characteristic function of a compound Poisson process, and hence* $\int_A f(x)N(t,dx)$ *is a compound Poisson process.*

3. Lévy-Itô decomposition

Theorem

(Lévy-Khintchine): For a Lévy process $L(t)$, its characteristic function is given by

$$\phi_{L(t)}(\theta) = \exp\left(t\left(ai\theta - \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}\setminus\{0\}}\left(e^{i\theta x} - 1 - i\theta x 1_{|x|<1}\right)\Pi(dx)\right)\right)$$

For some $\sigma-$finite measure Π called the **Lévy measure** of L satisfying

$$\int_{\mathbb{R}\setminus\{0\}} \min(1,x^2)\Pi(dx) < \infty$$

One should immediately recognise the similarity between characteristic function of $L(t)$ given by the Lévy Khintchine formula and the characteristic functions of the Poisson integrals of a function f given in the previous theorem. Indeed, let us examine what type of process the third term in the Lévy-Khintchine representation corresponds to:

$$\exp\left(\int_{\mathbb{R}\setminus\{0\}}\left(e^{i\theta x} - 1 - i\theta x 1_{|x|<1}\right)\Pi(dx)\right)$$

Then define $\nu := \Pi|_{\mathbb{R}\setminus(-1,1)}$ to be the restriction of Π to $\mathbb{R}\setminus(-1,1)$. Similarly let $\mu := \Pi|_{(-1,1)\setminus\{0\}}$. We can see that the above expression is equal to:

$$\exp\left(\int_{\mathbb{R}}\left(e^{i\theta x} - 1\right)\nu(dx) + \int_{\mathbb{R}}(e^{i\theta x} - 1 - i\theta x)\mu(dx)\right)$$

Recall the theorem from section 2, and substitute $A = \mathbb{R}\setminus(-1,1)$, $B = (-1,1)$ and f_1, f_2 for the inclusion functions. We see that the above expression is simply the characteristic function of

$$\int_{\mathbb{R}\setminus(-1,1)} xN(t,dx) + \int_{(-1,1)} x\tilde{N}(t,dx)$$

So the Lévy Itô decomposition allows us to write any Lévy process explicitly using its associated Poisson and compensated Poisson measures:

$$L(t) = a(t) + B(t) + \int_{\mathbb{R}\setminus(-1,1)} xN(t,dx) + \int_{(-1,1)} x\tilde{N}(t,dx)$$

i.e. every Lévy process is the sum of a Brownian with drift, a compound Poisson process, and a compensated Poisson process (the name given to the process represented by the last integral).

4. Lévy stochastic integrals

Now that we've seen that the associated Poisson measures align with the Lévy-Itô decomposition of a Lévy process, it makes sense to define Lévy integrals with respect to the Poisson measure.

Definition

Let $E \in \mathcal{B}(\mathbb{R})^d$ and $0 < T < \infty$. Then let $\mathcal{P}$ denote the smallest σ-algebra such that all mappings $F : [0, T] \times E \times \Omega \to \mathbb{R}$ satisfying:

(1) for each $0 \leq t \leq T$ the mapping $(x, \omega) \mapsto F(t, x, \omega)$ is $\mathcal{B}(E) \otimes \mathcal{F}_t-$measurable.
(2) For each $x \in E$, $\omega \in \Omega$ the mapping $t \mapsto F(t, x, \omega)$ is left continuous

are measurable. Then $\mathcal{P}$ is called the predicable σ-algebra and $\mathcal{P}$-measurable mapping is called **predictable**.

Definition

Let $\mathcal{P}_2(T, E)$ be the set of all equivalence classes of mappings $F : [0, T] \times E \times \Omega \to \mathbb{R}$ which are equal almost everywhere with respect to $\rho \times P$ ($\rho : (0, t] \times A = t\mu(A)$) such that F is predictable and

$$P\left(\int_0^T \int_E |F(t, x)|^2 \rho(dt, dx) < \infty\right) = 1$$

For a given Borel set A in $\mathbb{R}\setminus\{0\}$ bounded below, if we let $P(t)$ be the compound Poisson process defined by $P(t) = \int_A xN(t, dx)$, then for a predictable mapping $K : [0, T] \times \Omega \to \mathbb{R}$ we define:

$$\int_0^T \int_A K(t, x)N(dt, dx) := \sum_{0 \leq u \leq T} K(u, \Delta P(u))\chi_A(\Delta P(u))$$

$$\int_0^T \int_A K(t, x)\tilde{N}(dt, dx) := \int_0^T \int_A K(t, x)N(dt, dx) - \int_0^T \int_A K(t, x)\nu(dx)dt$$

Similar to the Poisson integral, this integral measures aggregate jumps (with values in A) of associated compound Poisson process, with jump sizes augmented by K. Then given $F \in \mathcal{P}_2(T)$, $H \in \mathcal{P}_2(T, \mathbb{R} \setminus \{0\})$, K predictable, a Brownian B_s and an arbitrary Poisson measure (independent to B_s) N with compensator $\tilde{N}$, we can define a **Lévy-type stochastic** integral as:

$$Y(t) = Y(0) + \int_0^t G(s)ds + \int_0^t F(s)dB_s + \int_0^t \int_{|x|<1} H(s, x)\tilde{N}(ds, dx)$$

$$+ \int_0^t \int_{|x|\geq 1} K(s, x)\tilde{N}(ds, dx)$$

The Lévy integral arises as a special case of such a stochastic integral. Given a Lévy process L with associated measures $N(t, x)$ and $\tilde{N}(t, x)$, and letting $X = \sqrt{a}F, H = K = xX$, then we say $Y(t)$ is a **Lévy integral** and write:

$$dY(t) = X(t)dL(t)$$

Lévy integrals come with their own Itô's formula:

Theorem

(Itô's formula): If Y is a Lévy type stochastic integral of the above form then for each $f \in C^2(\mathbb{R})$, $t \geq 0$ we have almost surely:

$$\begin{aligned} f(Y(t)) - f(Y(0)) = &\int_0^t \partial_x f(Y(s-))dY_c(s) + \frac{1}{2}\int_0^t \partial_x^2 f(Y(s-))d\langle Y_c\rangle(s) \\ &+ \int_0^t \int_{|x|\geq 1} f(Y(s-) + K(s,x)) - f(Y(s-))N(ds,dx) \\ &+ \int_0^t \int_{|x|<1} f(Y(s-) + H(s,x)) - f(Y(s-))\tilde{N}(ds,dx) \\ &+ \int_0^t \int_{|x|<1} f(Y(s-) + H(s,x)) - f(Y(s-)) \\ &- H(s,x)\partial_x f(Y(s-))\nu(dx)ds \end{aligned}$$

We can transform this into a slightly more general form, that actually can be applied to a wider class (semi-martingales):

Theorem

(Itô's formula 2): If Y is a Lévy type stochastic integral of the above form then, for each $f \in C^2(\mathbb{R})$, $t \geq 0$, we have almost surely:

$$\begin{aligned} f(Y(t)) - f(Y(0)) = &\int_0^t \partial_x f(Y(s-))dY(s) + \frac{1}{2}\int_0^t \partial_x^2 f(Y(s-))d\langle Y_c\rangle(s) \\ &+ \sum_{0\leq s\leq t} [f(Y(s)) - f(Y(s-)) - \Delta Y(s)\partial_x f(Y(s-))] \end{aligned}$$

To prove this second formula from the first one is mostly a consequence of applying Taylor's formula.

5. Worked example

We apply this second Itô formula to find the stochastic differential of $\left(e^{Y(t)}\right)_{t\geq 0}$, where Y is as a Lévy type stochastic integral. We then find an adapted process Z such that $dZ(t) = Z(t-)dY(t)$.
We have$f(x) = e^x$, so $\partial_x f(x) = e^x = \partial_x^2 f(x)$. Using the second Itô formula,

$$\begin{aligned} e^{Y(t)} - e^{Y(0)} = &\int_0^t e^{Y(s-)}dY(s) + \frac{1}{2}\int_0^t e^{Y(s-)}d\langle Y_c\rangle(s) \\ &+ \sum_{0\leq s\leq t} \left(e^{Y(s)} - e^{Y(s-)} - \Delta Y(s)e^{Y(s-)}\right) \end{aligned}$$

By definition, $Y(s) = Y(s-) + \Delta Y(s)$, so we can simplify the sum term:

$$\begin{aligned} e^{Y(s)} - e^{Y(s-)} - \Delta Y(s) e^{Y(s-)} &= e^{Y(s-)} e^{\Delta Y(s)} - e^{Y(s-)} - \Delta Y(s) e^{Y(s-)} \\ &= e^{Y(s-)} \left(e^{\Delta Y(s)} - 1 - \Delta Y(s) \right) \end{aligned}$$

Then converting notation, we get

$$d\left(e^{Y(t)}\right) = e^{Y(t-)} dY(t) + \frac{1}{2} e^{Y(t-)} d\langle Y_c \rangle(t) + \sum_{0 \le s \le t} e^{Y(s-)} \left(e^{\Delta Y(s)} - 1 - \Delta Y(s) \right)$$

To finish off, we can see that if we set $Z(t)$ as below, it will satisfy the given SDE $dZ(t) = Z(t-)dY(t)$,

$$Z(t) = \exp\left(Y(t) - \frac{1}{2} \langle Y_c \rangle(t) \right) \prod_{0 \le s \le t} e^{-\Delta Y(s)} \left(1 + \Delta Y(s)\right)$$

We show this by looking at both the continuous and jump parts. We use the second Itô's formula with only the continuous part first,

$$\begin{aligned} dZ_c(t) &= d\left(\exp\left(Y(t) - \frac{1}{2} \langle Y_c \rangle(t) \right) \right) \\ &= \exp\left(Y(t-) - \frac{1}{2} \langle Y_c \rangle(t-) \right) \left(dY(t) - \frac{1}{2} d\langle Y_c \rangle(t) \right) + \frac{1}{2} \exp\left(Y(t-) - \frac{1}{2} \langle Y_c \rangle(t-) \right) d\langle Y_c \rangle(t) \\ &= \exp\left(Y(t-) - \frac{1}{2} \langle Y_c \rangle(t-) \right) dY(t) \\ &= Z_c(t-) dY(t) \end{aligned}$$

For the jump part, we do something similar,

$$\begin{aligned} \Delta Z(t) &= Z(t) - Z(t-) \\ &= \exp\left(Y(t) - \frac{1}{2} \langle Y_c \rangle(t) \right) \prod_{0 \le s \le t} e^{-\Delta Y(s)} \left(1 + \Delta Y(s)\right) - Z(t-) \\ &= \exp\left(Y(t-) + \Delta Y(t) - \frac{1}{2} \langle Y_c \rangle(t) \right) e^{-\Delta Y(t)} \prod_{0 \le s < t} e^{-\Delta Y(s)} \left(1 + \Delta Y(s)\right) - Z(t-) \\ &= \exp\left(Y(t-) - \frac{1}{2} \langle Y_c \rangle(t) \right) \prod_{0 \le s < t} e^{-\Delta Y(s)} \left(1 + \Delta Y(s)\right) - Z(t-) \\ &= Z(t-)(1 + \Delta Y(t)) - Z(t-) \\ &= Z(t-) \Delta Y(t) \end{aligned}$$

where in the second last line we use that $Y_c(t) = Y_c(t-)$ by almost sure continuity. Putting the continuous and jump parts together gives that $dZ(t) = Z(t-)dY(t)$ as required.

6. Financial Application

As we spoke about in our presentation, Lévy processes can be used in financial markets. They can model asset prices which often exhibit sudden jumps due to events like earnings announcements, macroeconomic news, or geopolitical events. The classic Black-Scholes model, which assumes continuous paths driven by Brownian motion, fails to capture these discontinuities. To address this, Merton introduced a jump diffusion model where

the stock price follows a process with both continuous and jump components [2].
Let $S(t)$ denote the stock price at time t. In Merton's model, the log of the stock price $Y(t) = \ln S(t)$ is modeled as a Lévy process incorporating both Brownian motion and a compound Poisson process.
In this example, we will compute the price of a European call option on a non-dividend-paying stock whose price follows the Merton jump diffusion model. The goal is to derive an explicit formula for the option price and understand how jumps affect the valuation.
Consider a European call option with strike price K and maturity T. The stock price $S(t)$ follows the SDE:

$$dS(t) = S(t-)\left[\mu dt + \sigma dB(t) + (e^U - 1)dN(t)\right]$$

Here, $U \sim \mathrm{N}(\gamma, \delta^2)$ represents the log of the jump sizes, independent of $B(t)$ and $N(t)$. We learned Girsanov theory and the risk neutral measure $\mathbb{Q}$ in the seminar phase. Under $\mathbb{Q}$, the drift term μ is adjusted to $r - \lambda k$, where r is the risk-free interest rate, and $k = \mathbb{E}_{\mathbb{Q}}[e^U - 1]$ is the expected relative jump size. Then under $\mathbb{Q}$, the SDE becomes

$$dS(t) = S(t-)\left[(r - \lambda k)dt + \sigma dB^{\mathbb{Q}}(t) + (e^U - 1)dN^{\mathbb{Q}}(t)\right]$$

With this setup, the price of a European call option is given by the discounted expected payoff under $\mathbb{Q}$

$$C(S(0), K, T) = e^{-rT}\mathbb{E}_{\mathbb{Q}}\left[\max(S(T) - K, 0)\right]$$

It can be shown that this expectation can be computed by conditioning on the number of jumps that occur up to time T [2]. The number of jumps $N^{\mathbb{Q}}(T)$ follows a Poisson distribution with parameter λT. Therefore, the option price can be expressed as

$$C(S(0), K, T) = \sum_{n=0}^{\infty} e^{-\lambda T}\frac{(\lambda T)^n}{n!}C_n$$

where C_n is the Black-Scholes price of the option when there are n jumps, and the adjusted parameters are $S_n = S(0)e^{n\gamma + \frac{1}{2}n\delta^2}$, $\sigma_n^2 = \sigma^2 + n\delta^2$. We quickly derive these adjusted parameters.
Once we condition on some n, and take logs we get

$$\ln(S(T)) = \ln S(0) + \left(r - \lambda k - \frac{1}{2}\sigma^2\right)T + \sigma B(T) + \sum_{i=1}^{n} U_i,$$

where $U_i \sim \mathrm{N}(\gamma, \delta^2)$ (same as U). Then we know that

$$\sum_{i=1}^{n} U_i \sim \mathrm{N}(n\gamma, n\delta^2) \text{ and } \sigma B(T) \sim \mathrm{N}(0, \sigma^2 T).$$

So

$$S_n \colon = \mathbb{E}(\ln(S(T))) = \ln(S(0)) + (r - \lambda k - \frac{1}{2}\sigma^2)T + n\gamma \text{ and } \sigma_n^2 \colon = \mathbb{V}(\ln(S(T))) = \sigma^2 + n\delta^2.$$

For each n, C_n is computed using the Black-Scholes formula with adjusted initial stock price S_n and volatility σ_n. Recall the Black-Scholes formula for a European call option is

$$C_n = S_n\Phi\left(\frac{\ln\left(\frac{S_n}{K}\right) + \left(r + \frac{1}{2}(\sigma_n)^2\right)T}{\sigma_n\sqrt{T}}\right) - Ke^{-rT}\Phi\left(\frac{\ln\left(\frac{S_n}{K}\right) + \left(r + \frac{1}{2}(\sigma_n)^2\right)T}{\sigma_n\sqrt{T}} - \sigma_n\sqrt{T}\right)$$

Putting everything together, the option price is then

$$C(S(0),K,T) = \sum_{n=0}^{\infty} e^{-\lambda T}\frac{(\lambda T)^n}{n!}\left[S_n\Phi\left(\frac{\ln\left(\frac{S_n}{K}\right)+\left(r+\frac{1}{2}(\sigma_n)^2\right)T}{\sigma_n\sqrt{T}}\right)\right.$$
$$\left. - Ke^{-rT}\Phi\left(\frac{\ln\left(\frac{S_n}{K}\right)+\left(r+\frac{1}{2}(\sigma_n)^2\right)T}{\sigma_n\sqrt{T}} - \sigma_n\sqrt{T}\right)\right]$$

To get a feel for how these jumps actually affect the price, we compute the price given up to 14 jumps (this is when the jumps start to become negligible in the case provided). We start with the following setup for pricing a call option on a stock $S(t)$:

- Current stock price: $S(0) = \$100$, with strike price $K = \$100$ and $T = 1$ year;
- Risk-free interest rate: $r = 5\%$, volatility: $\sigma = 20\%$;
- Expected number of jumps per year: $\lambda = 4$ (quarterly reports);
- Mean of (log) jump size: $\gamma = -0.1$, standard deviation of (log) jump size: $\delta = 1$.

First, we computed the expected relative jump size:

$$k = e^{\gamma+\frac{1}{2}\delta^2} - 1 = e^{0.4} - 1 \approx 1.4918$$

Under $\mathbb{Q}$, the drift term is adjusted to account for k, that is $\mu = r - \lambda k \approx -1.9172$. Now we go through the details for the first few jumps, and then just summarise after that.
For $n = 0$ jumps, $S_0 = S(0)e^0 = 100$ and $\sigma_0 = \sigma = 0.2$, so we can compute C_0:

$$C_0 = S_0\Phi\left(\frac{\ln\left(\frac{S_0}{K}\right)+\left(r+\frac{1}{2}(\sigma_0)^2\right)T}{\sigma_0\sqrt{T}}\right) - Ke^{-rT}\Phi\left(\frac{\ln\left(\frac{S_0}{K}\right)+\left(r+\frac{1}{2}(\sigma_0)^2\right)T}{\sigma_0\sqrt{T}} - \sigma_0\sqrt{T}\right)$$
$$\approx 0$$

This is weighted by $e^{-4} \approx 0.0183$. So the total contribution is 0.
For $n = 1$ jumps, $S_1 = S(0)e^{-0.1+\frac{1}{2}} = 149.18$ and $\sigma_1 = \sqrt{0.2^2+1} \approx 1.0198$, so we can compute C_1:

$$C_1 = S_1\Phi\left(\frac{\ln\left(\frac{S_1}{K}\right)+\left(r+\frac{1}{2}(\sigma_1)^2\right)T}{\sigma_1\sqrt{T}}\right) - Ke^{-rT}\Phi\left(\frac{\ln\left(\frac{S_1}{K}\right)+\left(r+\frac{1}{2}(\sigma_1)^2\right)T}{\sigma_0\sqrt{T}} - \sigma_1\sqrt{T}\right)$$
$$\approx 22.22$$

This is weighted by $e^{-4} \times 4 \approx 0.0733$. So the total contribution is ≈ 1.63.
To start, as n increases, we will see C_n increase because of the positive expected value of the jump. We will also see that the weight decreases exponentially, so eventually the contributions will become negligible. We present a table of all contributions up to $n = 14$ below in Table 1.

n	C_n ($)	Weight	Contribution ($)
0	0	0.0183	0
1	22.22	0.0733	1.63
2	98.88	0.1465	14.48
3	215.52	0.1953	42.10
4	384.98	0.1953	75.22
5	634.49	0.1563	99.22
6	1,003.15	0.1042	104.60
7	1,548.10	0.0595	92.05
8	2,360.12	0.0298	70.41
9	3,571.59	0.0132	47.22
10	5,371.55	0.0053	28.43
11	8,061.30	0.0019	15.53
12	12,071.73	0.0006	7.42
13	17,955.10	0.0002	3.76
14	26,965.44	0.0001	1.23
Total			**$603.88**

TABLE 1. Contributions at each n.

We can see from the table provided that the contribution changes substantially as the number of jumps being conditioned on changes. To see how this compares to the standard Black-Scholes model, we provide the call option price in the (assumed) continuous case. Using all the same values as before, we get

$$C_{BS} = S_0\Phi\left(\frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) - Ke^{-rT}\Phi\left(\frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} - \sigma\sqrt{T}\right)$$
$$\approx 10.42$$

This is a drastic difference between the two models. This difference in option prices between the models can be attributed to the presence of jumps in the Merton model. While the Black-Scholes model considers only continuous price changes, the Merton model incorporates sudden and significant price jumps, which introduce additional risk and potential for higher option payoffs - particularly with the expected jump being positive.

This comparison shows how important it is incorporate the jump structure with a Lévy process into the model if it is believed that the underlying asset is not strictly continuous.

References

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