Project: Kervaire Invariant One Problem: C_2 Equivariance and KR theory

Course: Stable Homotopy Theory

Lecturers: Dr Vigleik Angeltveit

Student: Tobin Carlton Van Buizen

The Kervaire invariant one problem is a long standing problem in algebraic topology. This was (mostly) resolved by Hill, Hopkins and Ravenel using C_8 equivariance - showing how powerful equivariant stable homotopy theory can be when dealing with manifold invariants. This report looks at using C_2 equivariance, which is a simpler and well-understood step that is crucial before tackling the more complex C_8 equivariance used in the original solution. Using C_2 equivariance allows us to compute the homotopy type of real K-theory. Even though the C_2 approach is easier, it still gives us important results, like a gap theorem and a periodicity theorem. These methods used are key stepping stones that help us understand and eventually deal with the more complicated C_8 equivariant case.
In this report, we'll explain the methods used for the C_2 case, describe the computational techniques involved, and talk about what these findings mean. In particular, we will use the methods to compute the homotopy type of a K theory called KR theory.

1. Equivariant Homotopy Theory Basics

First, we start by introducing the basic definitions and theorems in equivariant homotopy theory.

Definition 1.1. For a topological group G, a G-space X is a topological space X together with a continuous map $G \times X \to X$ by $(g, x) \mapsto gx$.
A continuous map $f : X \to Y$ of G-spaces is a G-equivariant map if $f(gx) = gf(x)$ for all $g \in G$ and $x \in X$.

Definition 1.2. Let G be a topological group. If X, Y are G-spaces, we are interested in studying homotopy classes of equivariant maps between X and Y. Let $f, g : X \to Y$, a homotopy between f and g is a G-equivariant map $H : X \times I \to Y$, where I is given the trivial action, such that $H|_{X\times\{0\}} = f$ and $H|_{X\times\{1\}} = g$.

We want to translate the notion of a CW complex to the equivariant world. Eventually this will help with computing G-cellular homology and cohomology.

Definition 1.3. A G-CW structure on a G-space space X is given by a CW-structure where the cells are permuted by the action of G. That is, the n-skeleton is obtained from the $(n-1)$-skeleton by attaching $I \times D^n$ along $I \times \partial D^n$, where I is a G-set and the action of G on D^n is trivial.

Definition 1.4. For a G-space X, for each subgroup $H \subset G$ we have a fixed point space

$$X^H = \{x \in X : hx = x \text{ for all } h \in H\}$$

Definition 1.5. A genuine G-spectra is given by an orthogonal spectra equipped with a G action.

From now, X can be a G-space or a genuine G-spectrum, unless specifically stated.

Definition 1.6. We have equivariant homotopy groups $\pi_*^H X := \pi_* X^H$, for each $H \subseteq G$.

With these definitions, we use the following theorem as motivation for studying equivariant homotopy theory.

Theorem 1.7 (Equivariant Whitehead Theorem). A map $f : X \to Y$ of G-CW complexes is a G equivalence (G-equivariant homotopy equivalence) if and only if for all n and $H \subseteq G$,

$$\pi_n^H(X) \xrightarrow{\cong} \pi_n^H(Y)$$

As well as a Whitehead theorem, we get all the other lovely things from non-equivariant homotopy theory that we could ever want, perhaps most importantly, the long exact sequence from a fibration.
So with equivariant homotopy theory we recover all the non-equivariant goodness, and now we want to give even further structure, exhibiting why equivariant is better.

Definition 1.8. Let $\mathcal{F}^G$ denote the category of finite G sets. A Mackey functor $\underline{M}$ for a finite group G is a pair of functors $M_* : \mathcal{F}^G \to Ab$ and $M^* : (\mathcal{F}^G)^{op} \to Ab$ that agree on objects, send finite disjoint unions to direct sums, and such that for every pullback diagram in $\mathcal{F}^G$

$$\begin{array}{ccc} W & \xrightarrow{\alpha} & X \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\gamma} \\ Y & \xrightarrow[\delta]{} & Z \end{array}$$

we have $M_*(\gamma) \circ M^*(\delta) = M^*(\alpha) \circ M_*(\beta)$.

Example 1.9. If $G = C_p$, then the constant $\underline{\mathbb{Z}}$ Mackey functor is given by $\begin{array}{c} \mathbb{Z} \\ {\scriptstyle 1}\big\downarrow\ \big\uparrow{\scriptstyle p} \\ \mathbb{Z} \end{array}$

Lemma 1.10. For G a finite group, then genuine G-spectra are equivalent to Mackey functors on the category of finite G-sets.

Definition 1.11. We define Mackey functor homotopy groups, denoted $\underline{\pi}_* X$, by

$$\underline{\pi}_* X(G/H) := \pi_*^H X$$

Definition 1.12. For a Mackey functor $\underline{M}$, define Eilenberg-MacLane G-spaces, denoted by $K(\underline{M}, n)$, to be such that

$$\underline{\pi}_i(K(\underline{M}, n)) = \begin{cases} \underline{M} & i = n \\ 0 & \text{else} \end{cases}$$

Definition 1.13. Given a Mackey functor $\underline{M}$, there is an equivariant Eilenberg-MacLane spectrum $H\underline{M}$ such that

$$\underline{\pi}_n(H\underline{M}) = \begin{cases} \underline{M} & n = 0 \\ 0 & \text{else} \end{cases}$$

Once we have an Eilenberg-Maclane spectrum, we can define a homology and cohomology theory.

Definition 1.14. Given a Mackey functor $\underline{M}$, define equivariant homology with coefficients in $\underline{M}$ as

$$\underline{H}^G_*(X; \underline{M}) := \underline{\pi}^G_*(H\underline{M} \wedge X)$$

Define equivariant cohomology as

$$H^k_G(X; \underline{M}) := [X, \Sigma^k H\underline{M}]^G$$

Note 1.15. This is not defined by $\underline{H}_* X(G/H) = H_* X^H$.

Finally, with all these new tools, we can do similar things but with representations of the finite group we are working with.

Recall 1.16. Let $\mathbb{F}$ be a field, G a finite group (we only care about C_2), and V a finite-dimensional vector space over $\mathbb{F}$. A representation of G on V is a group homomorphism $\rho_V : G \to GL_{\mathbb{F}}(V)$. We write ρ_g for $\rho_V(g)$ or sometimes just V.
We are only considering finite dimensional, real representations. Further, we mainly use orthogonal representations, so V is a Hilbert space and the map ρ maps into $O(V)$.
We write $RO(G)$ for the real orthogonal representations of G.

2. Slice Spectral Sequence

With the basics of equivariant homotopy theory done, now we can work towards defining the slice spectral sequence.

For C_2, there are only two irreducible representation, both one-dimensional: the trivial representation (denoted by 1 or $\mathbb{R}$), which sends both elements to 1 and the sign representation (denoted by σ or $\mathbb{R}_-$) which sends the identity to 1 and the non-identity to -1. Therefore, any C_2-representation can be written as $\mathbb{R}^p \oplus (\mathbb{R}_-)^q$ for $p, q \in \mathbb{Z}$. Sometimes the following notation can be found in the literature $S^V = S^{p+q,q}$, where the first index is the topological dimension of the space. This notation is used in the proof of Theorem 2.9 for convenience.

Definition 2.1. $\rho := 1 \oplus \sigma$ is the regular representation for $G = C_2$.

Definition 2.2. If V is a G-representation, define $S^V = V \cup \{\infty\}$ (one point compactification of V), where G acts trivially on ∞, which is taken to be the basepoint. We write $\Sigma^V X = X \wedge S^V$.

Definition 2.3. We can define equivariant homotopy groups graded on representations

$$\pi_V^H(X) := [S^V \wedge G/H_+, X]$$

Definition 2.4. Let V be a G-rep, $\underline{M}$ a Mackey functor. We get an Eilenberg-MacLane space denoted $K(\underline{M}, V)$. This is a based G-space that is G-homotopic to a G-CW complex, with

$$\underline{\pi}_{V+k}^H(K(\underline{M}, V)) = \begin{cases} M(G/H) & k = 0, H \subseteq G \\ 0 & \text{else} \end{cases}$$

Now, in order to define the slice spectral sequence, we first need a filtration. We can extend the idea of a Postnikov tower, giving an equivariant analogue.

Recall 2.5. Non-equivariantly we have the Postnikov tower of KU, with Eilenberg-MacLane spectrum as fiber.

$$\begin{array}{ccc} \Sigma^{2n} H\mathbb{Z} & \longrightarrow & P^{2n}KU \\ & & \downarrow \\ & & P^{2n-2}KU \end{array}$$

Definition 2.6. Let V be a G-rep, define $S_V = \{S^W \wedge G/H_+ | W \supseteq V + 1, H \subseteq G\}$. Now, for a given G-space Y, define another G-space F_{S_V} as the pushout

$$\begin{array}{ccc} \coprod_\sigma \Sigma^n S & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \coprod_\sigma C\Sigma^n S & \dashrightarrow & F_{S_V} Y \end{array}$$

Then if the nth composition of F_{S_V} is $F_{S_V}^n$, we get inclusions

$$X \hookrightarrow F_{S_V} X \hookrightarrow F_{S_V}^2 X \hookrightarrow \dots$$

Define the equivariant Postnikov section to be $P^V X = \varinjlim F_{S_V}^n X$.

Theorem 2.7. Let X be a based C_2-space and V a C_2-rep. Then the map $X \mapsto P^V(X)$ induces an isomorphism $\pi_k^H(X) \to \pi_k^H(P^V(X))$ for $0 \leq k \leq \dim V^H$.

This tells us that this definition lines up with our knowledge of the non-equivariant Postnikov sections. We will write P^{2n} for the representation $n\rho$.

Eventually the main focus of this paper is KR-theory, so at some point it should be defined.

Definition 2.8 KR-theory is a variant of K-theory for spaces with an action of C_2. KR-theory is represented by an Ω-spectrum whose spaces are all equal to $\mathbb{Z} \times BU$.
The underlying non-equivariant spectrum of KR is equivalent to KU, the spectrum of G-fixed points of KR is equivalent to KO. We can think of this as the following Mackey functor:

$$\begin{array}{c} KO \\ {\scriptstyle C}\big\downarrow \big\uparrow{\scriptstyle R} \\ KU \end{array}$$

where the restriction map C is complexification which turns a real bundle into an associated complex bundle. The transfer map R is realification - we think of this in terms of vector spaces, but makes the easy analogue to bundles: R takes a complex vector space of dimension n and gives the real vector space of dimension $2n$.

Theorem 2.9. The homotopy fiber of $P^{2n}KR \to P^{2n-2}KR$ is homotopy equivalent to $\Sigma H\underline{\mathbb{Z}}$.

This means that equivariantly, we have the slice tower:

$$\begin{array}{ccc} \Sigma^{n\rho} H\underline{\mathbb{Z}} & \longrightarrow & P^{2n}K\mathbb{R} \\ & & \big\downarrow \\ & & P^{2n-2}K\mathbb{R} \end{array}$$

Proof. We show this level-wise. The homotopy fiber of $p^{2n}(\mathbb{Z} \times BU) \to P^{2n-2}(\mathbb{Z} \times BU)$ is homotopic to $K(\underline{\mathbb{Z}}, n\rho)$. Let F be the fiber. First, note that $\pi_k^G(P^{2n}(\mathbb{Z} \times BU)) \cong \pi_k^G(X) \cong \pi_k^G(P^{2n-2}(\mathbb{Z} \times BU))$ for $k \leq n-1$, and $\pi_n^G(P^{2n-2}(\mathbb{Z} \times BU)) = 0$. Then by the long exact sequence of equivariant homotopy groups, we get $\pi_k^G(F) = 0$ for $k \leq n-1$. Next, notice that $\pi_k^e = \pi_k$ so $\pi_k^e(F) = 0$ for $k \leq 2n-1$. Then by the definition of P^{2n}, $[S^{2n+j,n+k} \wedge G/H_+, P^{2n}(\mathbb{Z} \times BU)]_G = 0$ for all $j > 0, k \geq 0$ and $H \subset G$.
Finally, using the long exact sequence of homotopy groups again, we get that $\underline{\pi}_{2n}^{\rho}(F) = \underline{\mathbb{Z}}$. ■

Definition 2.10. The slice spectral sequence is the spectral sequence associated to the tower of fibrations $\{P^nX\}$ and it takes the form

$$E_2^{s,t} = \pi_{t-s}^G P^t X \implies \pi_{t-s}^G X$$

We could also apply the Mackey functor valued functor $\underline{\pi}$, giving

$$E_2^{s,t} = \underline{\pi}_{t-s}^G P^t X \implies \underline{\pi}_{t-s}^G X$$

The second superscript, the integer t, can be replaced by an element $V \in RO(G)$. This gives the $RO(G)$ graded spectral sequence which deserves its own definition.

Definition 2.11. We have the $RO(G)$ graded slice spectral sequence

$$E_2^{s,V} = \underline{\pi}^G_{V-s} P^V X \implies \underline{\pi}^G_{V-s} X$$

Note 2.12. The slice spectral sequence follows the Adams grading - so the homotopy in each dimension is given by some extension of the column of the E_∞ page.

3. Homology Computation

Using $G = C_2$, we will compute the equivariant homology of S^V with coefficients in $\underline{M}$. To do this, we compute using cellular equivariant homology.

Note 3.1. From our definitions, we note the following

$$\underline{\pi}_*(S^{n\rho} \wedge H\underline{\mathbb{Z}}) = \underline{H}_*(S^{n\rho}; \underline{\mathbb{Z}}) = \underline{H}_{*-n}(S^{n\sigma}; \underline{\mathbb{Z}})$$

Therefore, we can compute the homotopy groups by first computing (G-cellular) homology.

Definition 3.2. Let X be a G-CW complex with cellular chain complex C_*X. The latter is a chain complex of $\mathbb{Z}[G]$-modules, so we can apply Mackey functors and get a chain complex of Mackey functors, denoted by $\underline{C}_*X$. We then compute homology in the normal way, denoted by $\underline{H}_*X$. We can do the analogous thing for cochains and cohomology.

Note 3.3. We can recover non-equivariant homology $\underline{H}_*X(G/e) = H_*X$, and cohomology $\underline{H}^*X(G/e) = H^*X$.

Notation 3.4. We use notation $\mathbb{Z} = \mathbb{Z}[C_2]/(g-1)$ which means so $a + bg \equiv a + b$ in $\mathbb{Z}$ and $\mathbb{Z}_- = \mathbb{Z}[C_2]/(g+1)$ which means $a + bg \equiv a - b$ in $\mathbb{Z}_-$.

Notation 3.5. For C_2 we have the following Mackey functors:

$$\blacksquare : \begin{array}{c}\mathbb{Z}\\ {}_1\big\downarrow\big\uparrow{}_2\\ \mathbb{Z}\end{array} \quad \square : \begin{array}{c}0\\ \big\downarrow\big\uparrow\\ \mathbb{Z}_-\end{array} \quad \overline{\blacksquare} : \begin{array}{c}\mathbb{Z}\\ {}_2\big\downarrow\big\uparrow{}_1\\ \mathbb{Z}\end{array} \quad \bullet : \begin{array}{c}\mathbb{Z}/2\\ \big\downarrow\big\uparrow\\ 0\end{array} \quad \hat{\square} : \begin{array}{c}\mathbb{Z}/2\\ {}_0\big\downarrow\big\uparrow{}_1\\ \mathbb{Z}_-\end{array}$$

Lemma 3.6. We have short exact sequences
$0 \to \overline{\blacksquare} \to \blacksquare \to \bullet \to 0$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 0 \\
 & & {}_2\big\downarrow\big\uparrow{}_1 & & {}_1\big\downarrow\big\uparrow{}_2 & & \big\downarrow\big\uparrow & & \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow[1]{} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

$0 \to \bullet \to \hat{\square} \to \square \to 0$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}/2 & \xrightarrow{1} & \mathbb{Z}/2 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow\uparrow & & {\scriptstyle 0}\downarrow\uparrow{\scriptstyle 1} & & \downarrow\uparrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}_- & \xrightarrow[1]{} & \mathbb{Z}_- & \longrightarrow & 0
\end{array}$$

Proof. This is clear. ■

Definition 3.7. We call Δ the diagonal map, and ∇ the fold map where:

$$\Delta : \mathbb{Z} \to \mathbb{Z}[C_2], n \mapsto n(e+g)$$

$$\nabla : \mathbb{Z}[C_2] \to \mathbb{Z}, (ae+bg) \mapsto a+b$$

Theorem 3.8. Let C_2 have generator g, and consider $S^{n\sigma}$ (so n dimensions in the sign representation). We then have a reduced cellular chain complex C with

$$C_i^{n\sigma} = \begin{cases} \mathbb{Z}[C_2]/(g-1) & i = 0 \\ \mathbb{Z}[C_2] & 0 < i \leq n \\ 0 & \text{else} \end{cases}$$

Let $c_i^{(n)}$ denote a generator of $C_i^{n\sigma}$, then we can define the boundary map

$$d(c_{i+1}^{(n)}) = \begin{cases} c_i^{(n)} & i = 0 \\ g_{i+1-n}(c_i^{(n)}) & 0 < i \leq n \\ 0 & \text{else} \end{cases}$$

where $g_i = 1 - (-1)^i g$. Also, let $\epsilon_n = \begin{cases} 2 & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$

For $n \geq 0$, we compute cellular homology. We get the following diagram for $\underline{C}_* X$.

$$\begin{array}{ccccccccccc}
0 & & 1 & & 2 & & 3 & & & & n \\
\mathbb{Z} & \xleftarrow{2} & \mathbb{Z} & \xleftarrow{0} & \mathbb{Z} & \xleftarrow{2} & \mathbb{Z} & \longleftarrow & \cdots & \xleftarrow{\varepsilon_n} & \mathbb{Z} \\
{\scriptstyle 1}\downarrow\uparrow{\scriptstyle 2} & & {\scriptstyle \Delta}\downarrow\uparrow{\scriptstyle \nabla} & & {\scriptstyle \Delta}\downarrow\uparrow{\scriptstyle \nabla} & & {\scriptstyle \Delta}\downarrow\uparrow{\scriptstyle \nabla} & & & & {\scriptstyle \Delta}\downarrow\uparrow{\scriptstyle \nabla} \\
\mathbb{Z} & \xleftarrow[\nabla]{} & \mathbb{Z}[C_2] & \xleftarrow[g_2]{} & \mathbb{Z}[C_2] & \xleftarrow[g_3]{} & \mathbb{Z}[C_2] & \longleftarrow & \cdots & \xleftarrow[g_n]{} & \mathbb{Z}[C_2]
\end{array}$$

Now we compute kernel mod image which passes to homology.

$$
\begin{array}{ccccccccccc}
0 & & 1 & & 2 & & 3 & & & & n \\
\\
\mathbb{Z}/2 & \longleftarrow & 0 & \longleftarrow & \mathbb{Z}/2 & \longleftarrow & 0 & \longleftarrow & \cdots & \longleftarrow & \underline{H}_n(C_2/C_2) \\
\downarrow\uparrow & & \downarrow\uparrow & & \downarrow\uparrow & & \downarrow\uparrow & & & & \downarrow\uparrow \\
0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & \cdots & \longleftarrow & \mathbb{Z}_\pm
\end{array}
$$

where

$$
\underline{H}_n(C_2/C_2) = \begin{cases} \mathbb{Z} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}, \qquad \underline{H}_n = \begin{cases} \blacksquare & n \text{ even} \\ \square & n \text{ odd} \end{cases}, \qquad \mathbb{Z}_\pm = \begin{cases} \mathbb{Z} & n \text{ even} \\ \mathbb{Z}_- & n \text{ odd} \end{cases}
$$

This means for all $n \geq 0$, we have computed $\underline{H}_i(S^{n\sigma})$.
Next, for $n \leq -1$, we compute cellular cohomology in the absolute value of the dimensions (giving homology of the negative dimensions). We get the following diagram for $\underline{C}^* X$.

$$
\begin{array}{ccccccccccc}
0 & & 1 & & 2 & & 3 & & & & -n \\
\\
\mathbb{Z} & \xrightarrow{1} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \longrightarrow & \cdots & \xrightarrow{\varepsilon_{|n|}} & \mathbb{Z} \\
1\downarrow\uparrow 2 & & \Delta\downarrow\uparrow\nabla & & \Delta\downarrow\uparrow\nabla & & \Delta\downarrow\uparrow\nabla & & & & \Delta\downarrow\uparrow\nabla \\
\mathbb{Z} & \xrightarrow[\Delta]{} & \mathbb{Z}[C_2] & \xrightarrow[g_2]{} & \mathbb{Z}[C_2] & \xrightarrow[g_3]{} & \mathbb{Z}[C_2] & \longrightarrow & \cdots & \xrightarrow[g_{|n|}]{} & \mathbb{Z}[C_2]
\end{array}
$$

Now we compute kernel mod image which passes to cohomology.

$$
\begin{array}{ccccccccccccccc}
0 & & 1 & & 2 & & 3 & & 4 & & 5 & & & & -n \\
\\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & \cdots & \longrightarrow & \underline{H}^{|n|}(C_2/C_2) \\
\downarrow\uparrow & & \downarrow\uparrow & & \downarrow\uparrow & & \downarrow\uparrow & & \downarrow\uparrow & & \downarrow\uparrow & & & & \downarrow\uparrow \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & \mathbb{Z}_\pm
\end{array}
$$

where

$$
\underline{H}^{|n|}(C_2/C_2) = \begin{cases} \mathbb{Z} & n \text{ even} \\ \mathbb{Z}/2 & n \text{ odd} \end{cases}, \qquad \underline{H}^{|n|} = \begin{cases} \overline{\blacksquare} & n \text{ even} \\ \hat{\square} & n \text{ odd} \end{cases}, \qquad \mathbb{Z}_\pm = \begin{cases} \mathbb{Z} & n \text{ even} \\ \mathbb{Z}_- & n \text{ odd} \end{cases}
$$

This means that now we have computed $\underline{H}_i(S^{n\sigma})$ for all n. Finally, this gives

$$\underline{\pi}_i \Sigma^{n\rho} H\underline{\mathbb{Z}} = \begin{cases} \blacksquare & n \geq -1, n \text{ even}, i = 2n \\ \square & n \geq -1, n \text{ odd}, i = 2n \\ \overline{\blacksquare} & n \leq -2, n \text{ even}, i = 2n \\ \hat{\square} & n \leq 2, n \text{ odd}, i = 2n \\ \bullet & n \geq -1, n \leq i < 2n, i + n \text{ even} \\ \bullet & n \leq -2, 2n < i \leq n - 3, i + n \text{ odd} \\ 0 & \text{else} \end{cases}$$

4. Main Results

All of this preliminary work leads us to the main few theorems of the report. In the case of $G = C_2$ we get the Gap theorem and the Periodicity theorem. In the case of $G = C_8$ we would get these two theorems, as well as the Detection theorem, which gives the solution to Kervaire Invariant one.

From section 2, we can restate theorem 2.9 in the following way: the slices of KR are

$$P^n KR = \begin{cases} \Sigma^{(n/2)\rho} H\underline{\mathbb{Z}} & n \text{ even} \\ * & \text{else} \end{cases}$$

For the connective version kr, the slices are the same in nonnegative dimensions, but contractible in negative dimensions. Thus, we know the E_2 term for the wedge of all of the slices of KR and KR itself actually have the same E_2 term. Now combining this and all of our work gives the next theorem.

Theorem 4.1 The E_3 page of the slice spectral sequence with $G = C_2$ is shown in Figure 1.

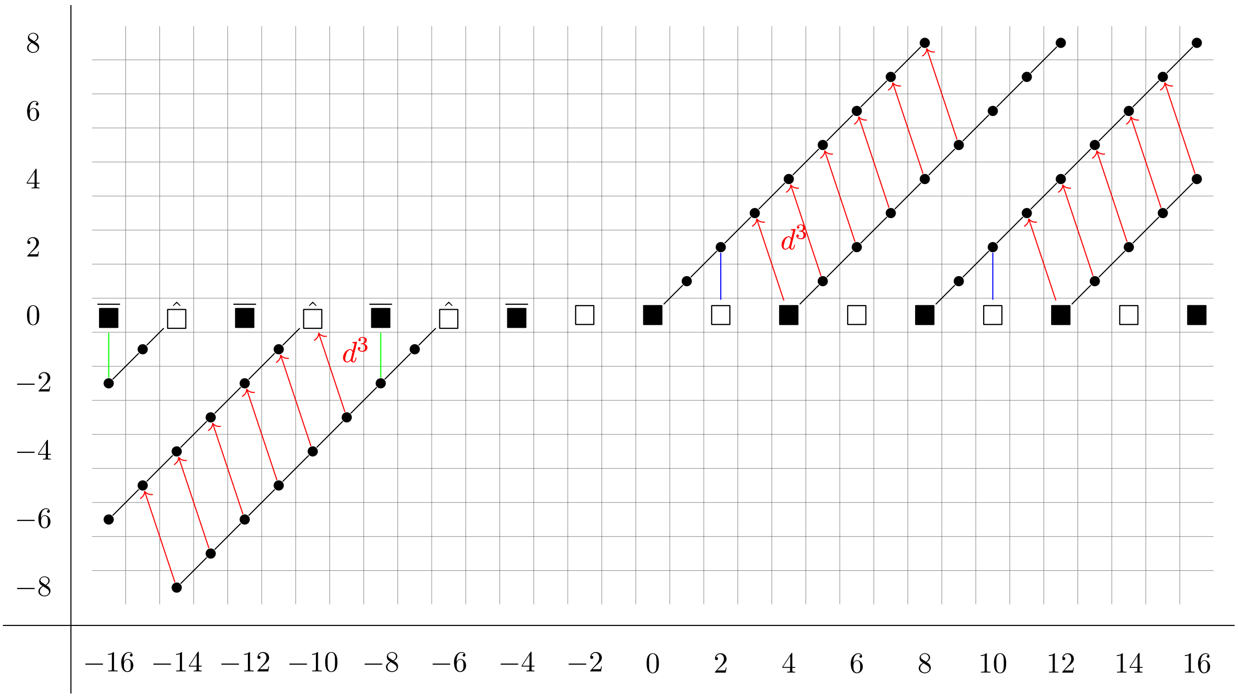

Figure 1: E_3 page.

Proof. From our hard work in section 3, we know this is how the the page should look, sans differentials, multiplications and extensions.
To justify the differentials, note that the slope 1 line is multiplication by η, the generator of π_1^s. Then, from the Adams spectral sequence of the sphere spectrum, we know that $\eta^4 = 0$, so this means that the $\bullet \in E^{4,4}$ must die, so we get differential from $E^{5,1}$. To justify the $\bullet \in E^{3,3}$ dying, we need to appeal to one of: Bott periodicity, homotopy groups of KU and KU or homotopy groups of BO. Unfortunately, there is nothing better than this to be found in the literature. The subsequent differentials follow by the multiplicative structure in the spectral sequence.

Recall the short exact sequences of Mackey functors from Lemma 3.6, these are the green and blue lines, giving so called "exotic" restrictions and transfers.

∎

Theorem 4.2 This spectral sequence computes the homotopy groups of KU and KO.

Proof. Since the non-equivariant spectrum of KR is KU, taking the bottom levels gives the homotopy groups of KU, with a $\mathbb{Z}$ in each even degree.
Since the G-fixed points of KR is KO, taking the top levels gives the homotopy groups of KO,

$$\pi_i KO = \begin{cases} \mathbb{Z} & i = 0 \mod 4 \\ \mathbb{Z}/2 & i = 1, 2 \mod 8 \\ 0 & i = 3, 5, 6, 7 \mod 8 \end{cases}$$

■

Theorem 4.3 (Gap theorem for C_2). $\pi_i(KR) = 0$ for $-4 < i < 0$.

As a consequence of this theorem, the H-equivariant groups for nontrivial H always vanish in dimensions strictly between -4 and 0.

Proof. We look at the columns $-4 < i < 0$ on the page above. Looking at $\square \in E_2^{0,-2}$, recall that the top level of $\square$ is 0. Everything else is empty, so the result is seen on the page of the spectral sequence. Note that this is best possible too, since both $\blacksquare$ and $\overline{\blacksquare}$ in $E^{0,0}$ and $E^{0,-4}$ respectively both are nontrivial on the top level. ■

Theorem 4.4 (Periodicity theorem for C_2). KR is 8-periodic, meaning that $\pi_i(KR)$ depends only on the reduction of i modulo 8. We have an equivalence $\Omega^8 KR \simeq KR$. This is Bott periodicity.

Proof. Looking at the top levels, we see that KO is 8 periodic. Looking at the bottom levels, we see that KU is 2 periodic. This means that KR is 8 periodic. ■

The astute reader will notice that the spectral sequence as presented is not $RO(C_2)$-graded. To get this, we can tensor the page presented with a particular module generated by a generator of MU (complex cobordism). This reader is directed to [4, Section 8].

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