VBKT: ODD PRIMARY HOPF INVARIANT ONE

TOBIN CARLTON VAN BUIZEN

Abstract. We give a consolidated exposition of the Hopf invariant one problem. At the prime 2 we mention Adams' secondary–operation argument showing that a map $f\colon S^{2n-1} \to S^n$ with Hopf invariant ± 1 exists only for $n = 1, 2, 4, 8$. For an odd prime p we follow the Adams–Atiyah translation into K-theory and prove that the mod-p Hopf invariant vanishes as soon as $n > 1$. The unique surviving case $n = 1$ is realised by an explicit map $S^{2p} \to S^3$ that we construct in detail.

1. Introduction

The purpose of this report is to give a clear and self-contained account of two classical theorems about maps of Hopf invariant one. The first, due to J.F.Adams, asserts that a map $f\colon S^{2n-1} \to S^n$ with integral Hopf invariant ± 1 can exist only when n is $1, 2, 4$ or 8. The second, proved by Adams and Atiyah, concerns the odd primary Hopf invariant: it shows that, for any odd prime p, the mod-p Hopf invariant of a map $S^{2np} \to S^{2n+1}$ necessarily vanishes as soon as $n > 1$.

We begin in Section 2 by giving the definition of the Hopf invariant, both with integral coefficients and modulo 2. This section also contains definitions of Steenrod squares, since these cohomology operations are the basic ingredients in every subsequent argument.

The odd-prime story requires more elaborate machinery. In Section 3 we introduce Steenrod powers and the odd primary Hopf invariant. Then we assemble the necessary tools from complex K-theory to reproduce the Adams–Atiyah filtration argument that rules out maps of odd-primary Hopf invariant one whenever $n > 1$.

Although the theorem leaves open the special case $n = 1$, an example is indeed known. We construct an explicit map $S^{2p} \to S^3$ whose mod-p Hopf invariant equals one, using a combination of Moore spaces, lens spaces and the Serre spectral sequence. This construction also illustrates how the general obstruction breaks down in lowest dimension.

For completeness, Section 4 sketches the construction of Steenrod squares and powers via the fibration $S^\infty \times_{\mathbb{Z}/p} X^{\wedge p} \to L(\infty, p)$.

2. Hopf Invariant One and Steenrod Squares

First, we recall the Hopf invariant as we know it, then we provide a different veiwpoint.

Definition 2.1 (Hopf invariant)**.** Let $f\colon S^{2n-1} \to S^n$ and form its mapping cone $C_f := S^n \cup_f e^{2n}$. This has integral cohomology given by

$$H^k(C_f;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & k = 0, n, 2n \\ 0 & \text{otherwise.} \end{cases}$$

Fix generators $\alpha \in H^n$ and $\beta \in H^{2n}$. Since $\alpha^2 \in H^{2n}$ there is a unique integer $H(f)$ such that $\alpha^2 = H(f)\,\beta$. This integer, well–defined up to sign, is the Hopf invariant of f.

Note that changing the orientation of either sphere multiplies both α and β by -1, so $H(f)$ is stable under the choice of orientation.

Definition 2.2. For every space X, there are natural cohomology operations called Steenrod squares. These are homomorphisms $\mathrm{Sq}^i\colon H^n(X;\mathbb{Z}/2) \to H^{n+i}(X;\mathbb{Z}/2)$ characterised uniquely by the following properties:

1. Naturality, for $f\colon X \to Y$, $\mathrm{Sq}^i(f^*(x)) = f^*(\mathrm{Sq}^i(x))$
2. $\mathrm{Sq}^i(x+y) = \mathrm{Sq}^i(x) + \mathrm{Sq}^i(y)$
3. Cartan formula $\mathrm{Sq}^i(x \smile y) = \sum_k \mathrm{Sq}^k(x) \smile \mathrm{Sq}^{i-k}(y)$
4. $\mathrm{Sq}^i(\sigma(x)) = \sigma(\mathrm{Sq}^i(x))$ where $\sigma\colon H^m(X;\mathbb{Z}/2) \to H^{m+1}(\Sigma X;\mathbb{Z}/2)$ is the suspension isomorphism given by multiplication with the fundamental class of $H^1(S^1;\mathbb{Z}/2)$.
5. $\mathrm{Sq}^i(x) = \begin{cases} x^2 & i = |x| \\ 0 & i > |x| \end{cases}$
6. $\mathrm{Sq}^0 = \mathrm{id}$
7. Sq^1 is the $\mathbb{Z}/2$ Bockstein homomorphism β associated with the coefficient sequence $0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$.

Property (5) explains the name of these operations. Steenrod squares are stable operations (property (4)) extending the cup square.

Definition 2.3 (Mod 2 Hopf invariant)**.** Let $f\colon S^{2n-1} \to S^n$ with mapping cone C_f, and reduce α and β from Definition 2.1 modulo 2. Let $x_n \in H^n(C_f)$ and $x_{2n} \in H^{2n}(C_f)$ be the mod 2 reductions of α and β. Then $\mathrm{Sq}^n(x_n) = H_2(f)x_{2n} \in H^{2n}(C_f;\mathbb{Z}/2)$, and the integer $H_2(f) \in \{0,1\}$ is the mod 2 Hopf invariant. Notice that $H_2(f) = 1$ exactly when $\mathrm{Sq}^n\colon H^n(C_f;\mathbb{Z}/2) \to H^{2n}(C_f;\mathbb{Z}/2)$ is non-zero.

Theorem 2.4 (Adams)**.** *A map $f\colon S^{2n-1} \to S^n$ with $H_2(f) = 1$ can exist only for $n = 1, 2, 4, 8$.*

To prove this, Adams [1] introduces a secondary operation built from Sq^{2n-1} and shows that its indeterminacy vanishes unless n is a power of 2.

Theorem 2.5. *If $f\colon S^{2n-1} \to S^n$ has Hopf invariant one, then $[f] \in \pi^s_{n-1}$ is nontrivial and so $\Sigma^k f\colon S^{2n+k-1} \to S^{n+k}$ are all homotopically nontrivial.*

Proof. If $f\colon S^{2n-1} \to S^n$ has Hopf invariant one, then by the fifth property of Sq, $\mathrm{Sq}^n\colon H^n(C_f;\mathbb{Z}/2) \to H^{2n}(C_f;\mathbb{Z}/2)$ is nontrivial. Then by the fourth property, we similarly get $\mathrm{Sq}^n\colon H^{n+k}(\Sigma^k C_f;\mathbb{Z}/2) \to H^{2n+k}(\Sigma C_f;\mathbb{Z}/2)$ for each k. Assume for a contradiction that $\Sigma^k f$ were homotopically trivial, then there would be a retraction $r\colon \Sigma^k C_f \to S^{n+k}$. Then a diagram would commute by the naturality of Sq^n. But clearly $H^{2n+k}(S^{n+k};\mathbb{Z}/2) = 0$, a contradiction.

$$\begin{array}{ccc} H^{n+k}(S^{n+k};\mathbb{Z}/2) & \xrightarrow{r^*} & H^{n+k}(\Sigma^k C_f;\mathbb{Z}/2) \\ {\scriptstyle \mathrm{Sq}^n}\downarrow & & {\scriptstyle \mathrm{Sq}^n}\downarrow \\ H^{2n+k}(S^{n+k};\mathbb{Z}/2) & \xrightarrow{r^*} & H^{2n+k}(\Sigma^k C_f;\mathbb{Z}/2) \end{array}$$

□

3. Odd Primary

The integral and mod 2 story extends to any odd prime p, after replacing Steenrod squares by the Steenrod powers P^i.

Definition 3.1 (Steenrod powers)**.** For every space X, there are natural cohomology operations called Steenrod powers. These are homomorphisms $P^i\colon H^n(X;\mathbb{Z}/p) \to H^{n+2i(p-1)}(X;\mathbb{Z}/p)$, for odd primes p, characterised uniquely by the following properties:

(1) Naturality, for $f\colon X \to Y$, $P^i(f^*(x)) = f^*(P^i(x))$
(2) $P^i(x+y) = P^i(x) + P^i(y)$
(3) Cartan formula $P^i(x \smile y) = \sum_k P^k(x) \smile P^{i-k}(y)$
(4) $P^i(\sigma(x)) = \sigma(P^i(x))$ where $\sigma\colon H^m(X;\mathbb{Z}/p) \to H^{m+1}(\Sigma X;\mathbb{Z}/p)$ is the suspension isomorphism given by multiplication with the fundamental class of $H^1(S^1;\mathbb{Z}/p)$.
(5) $P^i(x) = \begin{cases} x^p & 2i = |x| \\ 0 & 2i > |x| \end{cases}$
(6) $P^0 = \mathrm{id}$

Theorem 3.2. *The Steenrod squares and powers satisfy the so-called Adem relations*

$$[\text{if } a < 2b] \qquad \mathrm{Sq}^a\,\mathrm{Sq}^b = \sum_{j=0}^{a+b} \binom{b-j-1}{a-2j} \mathrm{Sq}^{a+b-j}\,\mathrm{Sq}^j$$

$$[\text{if } a < pb] \qquad P^a P^b = \sum_{j=0}^{a+b} (-1)^{a+j} \binom{(p-1)(b-j)-1}{a-pj} P^{a+b-j} P^j$$

$$P^a \beta P^b = \sum_{j=0}^{a+b} (-1)^{a+j} \binom{(p-1)(b-j)-1}{a-pj} \beta P^{a+b-j} P^j$$

$$[\text{if } a \le pb] \qquad - \sum_{j=0}^{a+b} (-1)^{a+j} \binom{(p-1)(b-j)-1}{a-pj-1} P^{a+b-j} \beta P^j$$

We take $\binom{n}{r} = 0$ *if* n *or* r *is negative or if* $r < n$.

These can be proven from the construction of the Steenrod squares and Steenrod powers.

We use these new operations to define the main topic of this paper, odd primary Hopf invariant.

Definition 3.3 (Odd primary Hopf invariant)**.** Let $f\colon S^{2np} \to S^{2n+1}$ with mapping cone $C_f = S^{2n+1} \cup_f e^{2np+1}$. The composite

$$P^n\colon H^{2n+1}(C_f;\mathbb{Z}/p) \to H^{2np+1}(C_f;\mathbb{Z}/p)$$

is called the odd primary Hopf invariant of f.

Adams and Atiyah translated the odd primary problem into K-theory as well, where Adams operations act diagonally after rationalising. We now summarise the K-theoretic filtration argument of Adams-Atiyah in [2].

Theorem 3.4. *Let p be an odd prime, n a positive integer not dividing $p-1$ and $1 \leq i \leq p$. Then, for a suitable integer k, the p-primary factor of*

$$\prod_{1<j<p, i\neq j} (k^{ni} - k^{nj}) \leq p^{n-1}$$

In what is to come, it will be useful to filter $K^*(X)$. When X is a CW complex, we can filter $K^*(X)$ using the descending filtration

$$F_sK(X) = \ker(K(X) \to K(X^{s-1}))$$

where X^s is the s-skeleton of X.

Lemma 3.5. *If $x \in F_{2n}K(X)$, then $\psi^k(x) - k^n x \in F_{2n+1}K(X)$.*

Proof. From the long exact sequence of the pair (X^{2n}, X^{2n-1}), we can look at the below piece,

$$K(X^{2n}/X^{2n-1}) \xrightarrow{\pi^*} K(X^{2n}) \xrightarrow{i^*} K(X^{2n-1})$$

Take $x \in F_{2n}K(X)$. There is the natural inclusion $j\colon X^{2n} \hookrightarrow X$. We know that j^*x maps into the kernel of i^* in the sequence. So by exactness, we have $j^*x = \pi^*y$ for some $y \in K(X^{2n}/X^{2n-1})$. Then,

$$\begin{aligned} j^*(\psi^k - k^n)(x) &= (\psi^k - k^n)j^*x \\ &= (\psi^k - k^n)\pi^*y \\ &= \pi^*(\psi^k - k^n)y \end{aligned}$$

But, X^{2n}/X^{2n-1} is a wedge sum of S^{2n}s. Thus, $\psi^k(y) - k^n y = 0$. Therefore, $(\psi^k - k^n)x \in \ker j$. Hence, in $F_{2n+1}K(X)$. □

We can define the K-theory analogue of Betti numbers as follows:

$$B_{2n} = \dim_{\mathbb{Q}}(F_{2n}K(X) \otimes \mathbb{Q}/F_{2n+1}K(X) \otimes \mathbb{Q})$$

By the Chern character isomorphism, these are the same as the normal Betti numbers in cohomology.

Lemma 3.6. *Let X be a finite dimensional CW complex. Suppose $B_{2n}(X) = 0$ for $n \neq 0$, $n_1 < \cdots < n_r$. Let $k_0, ..., k_r \in \mathbb{Z}$, then*

$$\prod_{i=0}^{r} \left(\psi^{k_i} - k_i^{n_i}\right) = 0$$

in $\tilde{K}(X) \otimes \mathbb{Q}$.

Proof. All terms in the product commute. By Lemma 3.5, if $x \in F_{2n_i}K(X) \otimes \mathbb{Q}$, then $(\psi^{k_i} - k_i^{n_i})x \in F_{2n_i+1}K(X) \otimes \mathbb{Q}$. Using the assumption we get $(\psi^{k_i} - k_i^{n_i})x \in F_{2n_i+1}K(X)$. By repeating the argument, for every element of the product we get,

$$\prod_{i=0}^{r}(\psi^{k_i} - k_i^{n_i}) \in F_{2n_r+1}K(X)$$

Hence, $\prod_{i=0}^{r}(\psi^{k_i} - k_i^{n_i}) = 0$, because $F_{2n_r+1}K(X) = 0$ by assumption. □

Using this formula, we get an Eigenspace decomposition for ψ^k. We give an outline of how this works below.
Let $V_{i,k} = \operatorname{Im}\prod_{j\neq i}(\psi^k - k^{n_j}) \in \tilde{K} \otimes \mathbb{Q}$. Then from linear algebra, we have $V_{i,k} = \ker(\psi^k - k^{n_i})$ and $\bigoplus_i V_{i,k} = \tilde{K}(X) \otimes \mathbb{Q}$. By Lemma 3.6,

$$V_{i,k} = \operatorname{Im}\prod_{j\neq i}(\psi^k - k^{n_j}) \subset \ker(\psi^\ell - \ell^{n_i}) = V_{i,\ell}$$

Since k, ℓ were arbitrary, we have that $V_{i,k} = V_{i,\ell}$ for any k, ℓ. Thus, we can omit the second subscript, and have $\bigoplus_i V_i = \tilde{K}(X) \otimes \mathbb{Q}$. We have a projection map $\pi_i \colon \tilde{K}(X) \otimes \mathbb{Q} \to V_i$ given by

$$\pi_i(x) = \prod_{j\neq i}\frac{\psi^{k_j} - k_j^{n_j}}{k_j^{n_i} - k_j^{n_j}}$$

Note that $\pi_i x = x$ for $x \in V_i$ and $\pi_i x = 0$ for $x \in V_j$ for $j \neq i$. This follows from V_i being the eigenspaces of ψ^k. Decomposing into a sum $\sum_i \pi_i(x)$ will be useful in proofs to come.

Definition 3.7. Define $d_i(n_1, ..., n_r)$ to be the gcd of all products of the form $\prod_{j\in\{1,...,r\}\setminus\{i\}}(k_j^{n_i} - k_j^{n_j})$. Using the formula for $\pi_i(x)$, the denominator of $\pi_i(x)$ divides $d_i(n_1, ..., n_r)$.

Theorem 3.8. *Let X be a finite CW complex. Suppose $p^{n_i} \nmid d_i(n_1, ..., n_r)$ for each i. Then $\psi^p(x) \in p\tilde{K}(X) \oplus \operatorname{Torsion}(\tilde{K}(X))$.*

Proof. Can write any $x \in \tilde{K}(X) \otimes \mathbb{Q}$ as $\sum_i \pi_i(x)$. Then,

$$\psi^p(x) = \sum_i \psi^p(\pi_i(x)) = \sum_i p^{n_i}\pi_i(x)$$

If x is integral and p^{n_i} does not divide $d_i(n_1, ..., n_r)$, then each term in the sum must be of the form py_i/q_i with y_i integral and $p \nmid q_i$. Hence, the full sum is of the form py/q with $p \nmid q$. If we look at what this means in $\tilde{K}(X)$, we get $\psi^p(x) \in p\tilde{K}(X) \oplus \operatorname{Torsion}(\tilde{K}(X))$. □

Definition 3.9. We define $\mathbb{Z}$ localised at p, written $\mathbb{Z}_{(p)}$, to be

$$\mathbb{Z}_{(p)} = \left\{\frac{a}{b} \mid a \in \mathbb{Z}, b \in \mathbb{Z} \setminus \{0\}, p \nmid b\right\}$$

Corollary 3.10. *Let X be a finite CW complex. Suppose $p^{n_i} \nmid d_i(n_1, ..., n_r)$ for each i. Suppose further that $H^*(X;\mathbb{Z})$ has no p-torsion. Then the pth cup power is 0 mod p.*

Proof. To prove this, we use the Atiyah-Hirzebruch spectral sequence,

$$H^*(X; K^*(pt)) \Rightarrow K^*(X)$$

with all differentials as torsion operators. We can also tensor the spectral sequence with $\mathbb{Z}_{(p)}$. Then, since $H^*(X; K^*(pt))$ has no p torsion, the resulting spectral sequence has all differentials zero. Hence, $E_2 = E_\infty$ and

$$H^{2n}(X; \mathbb{Z}_{(p)}) \cong H^{2n}(X) \otimes \mathbb{Z}_{(p)} \cong K_{2n}(X)/K_{2n+1}(X) \otimes \mathbb{Z}_{(p)}$$

Since X has no p torsion, the map $H^*(X; \mathbb{Z}_{(p)}) \to H^*(X; \mathbb{Z}/p)$ is surjective. Now, any $y \in H^*(X; \mathbb{Z}/p)$ is in the image of some element $x \otimes a \in K(X) \otimes \mathbb{Z}_{(p)}$. But, $(x \otimes a)^p = x^p \otimes a^p = 0$ since $x^p = 0$ by Theorem 3.8. Hence, the pth power map is zero. □

Lemma 3.11. *Let p be an odd prime, n a positive integer with $n \nmid p-1$. Then there exists positive integer k such that p^n does not divide $\prod_{j \in \{1,...,p\} \setminus \{i\}} (k^{ni} - k^{nj})$ for all $i \in \{1, ..., p\}$.*

Proof. Let p^e be the highest power of p dividing

$$\prod_{j \in \{1,\dots,p\} \setminus \{i\}} (k^{ni} - k^{nj})$$

with k to be determined. We want to show that $e \leq n-1$. Suppose $\gcd(n, p-1) = h$, then $n = ah$ and $p - 1 = bh$ with $a, b > 1$ since $n \nmid p-1$. Let p^f be the highest power of p dividing n. Then $(\mathbb{Z}/p^{f+2})^\times$ is a cyclic group of order $p^{f+1}(p-1)$. Choose k such that it generates this group. Then k also generates $(\mathbb{Z}/p^\ell)^\times$ for $\ell < f+2$.

We want to get a formula for e. Note that the highest power of p dividing

$$\prod_{j \in \{1,\dots,p\} \setminus \{i\}} (k^{ni} - k^{nj})$$

is the product of the highest powers of p dividing $(k^{ni} - k^{nj})$. If $i < j$, $(k^{ni} - k^{nj}) = (k^{n(i-j)} - 1)k^{nj}$. Since p and k are coprime, the highest power of p dividing this term is the highest power of p dividing the first factor, $k^{n(i-j)} - 1$. Since k generates $(\mathbb{Z}/p^\ell)^\times$, p^ℓ divides $k^{n(i-j)}$ if and only if $p^{\ell-1}(p-1)$ divides $n(i-j)$. From this, we can determine what the highest power of p dividing each term in the product is. The argument works similarly for $i < j$.
Thus, we get

$$e = (f+1)\left(\left\lfloor \frac{i-1}{b} \right\rfloor + \left\lfloor \frac{p-i}{b} \right\rfloor\right)$$

Hence,

$$e \leq (f+1)\frac{p-1}{b} = h(f+1) \leq hp^f \leq n$$

But, equality cannot hold throughout, since if $p^f = f+1$, then f must be 0 and $hp^f = h < n$. Hence, $e < n$. □

Theorem 3.12. *Let p be an odd prime, n a positive integer with $n \nmid p-1$. Assume $H^*(X; \mathbb{Z})$ has no p torsion and $H^{2k}(X; \mathbb{Q}) = 0$ if $n \nmid k$. Then the cup pth power $H^{2n}(X; \mathbb{Z}/p) \to H^{2np}(X; \mathbb{Z}/p)$ is zero.*

This theorem is the key tool used in proving our main theorem.

Proof. Note that we can replace X by X^{2np+1} since we are only looking at cohomology. Then we know that all $B_{2k}(X)$ are zero except for possibly when k is a multiple of n. By Lemma 3.11 and Corollary 3.10, we have p^{n_i} does not divide $d_i(n_1, ..., n_r)$ so we conclude that the pth power is zero mod p. □

After all of this set up, we can finally state and prove the main theorem.

Theorem 3.13 ([2, Odd Primary Hopf Invariant One])**.** *For $n > 0$, the odd primary Hopf invariant is zero. That is,*

$$P^n : H^{2n+1}(C_f; \mathbb{Z}/p) \to H^{2np+1}(C_f; \mathbb{Z}/p)$$

is zero.

Proof. Since $n > 1$, either n divides $p-1$ or it does not. If it does, we use the Adem relations so that $P^n = \frac{1}{n}P^1P^{n-1}$. But $H^{2kn+1}(C_f)$ is zero for $1 < k < n$, so P^{n-1} is zero, which gives P^n is the zero map.

Now, assume $n \nmid p-1$. Let $f : S^{2np} \to S^{2n+1}$ be our map of spheres. Noting that $S^{2n+1} = \Sigma S^{2n}$, we use the Σ-Ω adjunction to get a map $g : S^{2np-1} \to \Omega S^{2n+1}$. From this, we will look at the mapping cone $C_g = \Omega S^{2n+1} \cup_g e^{2np}$. We then compute

$$H^{2kn}(\Omega S^{2n+1}) = \begin{cases} \mathbb{Z} & k \in \mathbb{N} \\ 0 & \text{else.} \end{cases}$$

To do this computation, we use the Serre spectral sequence with the pathspace fibration $\Omega S^{2n+1} \to PS^{2n+1} \to S^{2n+1}$. The $E_{2n-1} = E_\infty$ page can be seen below.

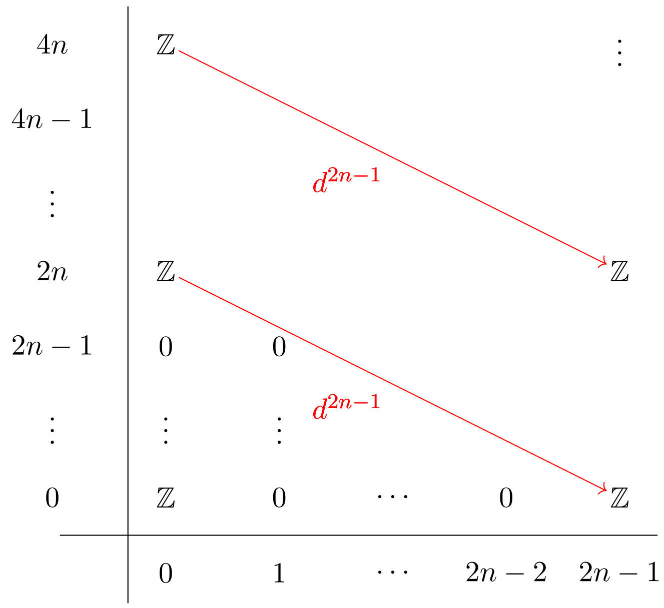

FIGURE 1. E_{2n-1} page.

From this, we just read off the cohomology as described.
Note that attaching a $2np$ cell does not change the even-degree groups in dimension less than $2np$, so $C_g = \Omega S^{2n+1} \cup_g e^{2np}$ has the same integral (hence mod p) cohomology pattern. So almost all conditions for Theorem 3.12 are satisfied, except that C_g is not a finite CW complex. This is easily fixed, because we can approximate C_g by a finite complex up to any dimension, so we get that the pth cup power $H^{2n}(C_g; \mathbb{Z}/p) \to H^{2np}(C_g; \mathbb{Z}/p)$ is zero, so after suspending, we have $P^n\colon H^{2n+1}(\Sigma C_g; \mathbb{Z}/p) \to H^{2np+1}(C_g; \mathbb{Z}/p)$ is also 0.

Then, by the definition of f, f is homotopic to the composite

$$S^{2np} \xrightarrow{\Sigma g} \Sigma\Omega S^{2n+1} \xrightarrow{e} S^{2n+1}$$

where $e(t, w) = w(t)$. This means that e extends the map $e'\colon \Sigma C_g \to C_f$. Since e induces an isomorphism on H^{2n+1}, we have that e' induces an injection in cohomology. This proves the result. □

Theorem 3.13 has $n > 1$ in the hypothesis, which raises the question whether an example can be found for $n = 1$. To give this example that does indeed exist, we need a bit more machinery.

Theorem 3.14 ([3, 4I.3]). *For any prime power p^n, $\Sigma K(\mathbb{Z}/p^n, 1)$ is homotopy equivalent to $X_1 \vee ... \vee X_{p-1}$ where X_i is a CW complex, with $\tilde{H}_k(X_i; \mathbb{Z}) \neq 0$ only when $k \equiv 2i \mod 2p-2$.*

Proof idea. This can be proven using the Atiyah-Segal completion theorem, along with Bott periodicity and Adams operations. □

Definition 3.15. We can build a CW complex X with $H_n(X) \cong G_n$ for all n by constructing inductively an increasing sequence of subcomplexes $X_1 \subset X_2 \subset \cdots$ with $H_i(X_n) \cong G_i$ for $i \leq n$ and $H_i(X_n) = 0$ for $i > n$ where

- X_1 is a Moore space $M(G_1, 1)$;
- X_{n+1} is the mapping cone of a cellular map $h^n\colon M(G_{n+1}, n) \to X_n$ such that the induced map $h^n_*\colon H_n(M(G_{n+1}, n)) \to H_n(X_n)$ is trivial;
- $X = \bigcup_n X_n$.

For a space Y, a homotopy equivalence $f\colon X \to Y$ where X is constructed as above, is called a homology decomposition of Y.

Theorem 3.16 ([3, 4H.3]). *Every simply connected CW complex has a homology decomposition.*

Proof idea. This can be proven using Hurewicz's theorem, note that a Moore space is a wedge of spheres, then use the five lemma on a pair of long exact sequence of pairs. □

Proposition 3.17 ([3, 4C.1]). *Let X be a simply connected CW complex with a decomposition of its homology groups $H_n(X)$ as a direct sum of cyclic groups with specified generators, then there is a CW complex Z and a CW homotopy equivalence $f\colon Z \to X$ such that each cell of Z is either:*

- *a generator n-cell e^n_α, which is a cycle in cellular homology, mapped by f to a cellular cycle representing the specified generator α of one of the cyclic summands of $H_n(X)$;*
- *a relator $(n+1)$-cell e^{n+1}_α, with cellular boundary equal to a multiple of the generator n-cell e^n_α, in the case that α has finite order.*

Proof idea. This can be proven by building Z inductively over skeleta, with Z^1 a point since X is simply connected. We then use the mapping cylinder M_f, use Hurewicz's theorem, and do a diagram chase. □

Example 3.18 ([3, 4L.6, A map with odd primary Hopf invariant one])**.** Let p be an odd prime. We will build a map

$$f\colon S^{2p} \longrightarrow S^3$$

whose mapping cone $C_f = S^3 \cup_f e^{2p+1}$ satisfies

$$P^1 : H^3(C_f;\mathbb{Z}/p) \to H^{2p+1}(C_f;\mathbb{Z}/p) \quad \text{is non-zero.}$$

As a result, we will have found a map f with odd primary Hopf invariant one. Note that this is important, since f represents a non-trivial class in the stable stem π^s_{2p-3}.

The group $H^2(K(\mathbb{Z}/p,1);\mathbb{Z}/p)$ is non-zero, and by property (5) for Steenrod powers $P^1(x) = x^p \neq 0$ for a generator $x \in H^2(K(\mathbb{Z}/p,1);\mathbb{Z}/p)$. By Theorem 3.14 we have,

$$\Sigma K(\mathbb{Z}/p,1) \simeq \bigvee_{i=1}^{p-1} X_i,$$

where each X_i has reduced homology

$$\tilde{H}_j(X_i;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/p & j \equiv 2i \mod 2p-2 \\ 0 & \text{else.} \end{cases}$$

Consider the summand $X = X_1$, then $H^3(X;\mathbb{Z}/p)$ and $H^{2p+1}(X;\mathbb{Z}/p)$ are both non-trivial, and

$$P^1\colon H^3(X;\mathbb{Z}/p) \to H^{2p+1}(X;\mathbb{Z}/p)$$

is non-zero because it is inherited from the wedge inclusion.

Then, since $\pi_1(X) = 0$, the construction in Proposition 3.17 realises the $(2p+1)$-skeleton of X as

$$C_g = S^2 \cup e^3 \cup e^{2p} \cup e^{2p+1},$$

Then, using the homology decomposition, we can take this skeleton to be the mapping cone C_g of a map of Moore spaces $g\colon M(\mathbb{Z}/p, 2p-1) \to M(\mathbb{Z}/p, 2)$. We then collapse the bottom S^2 of C_g,

$$h\colon M(\mathbb{Z}/p, 2p-1) \xrightarrow{g} M(\mathbb{Z}/p, 2) \twoheadrightarrow S^3.$$

The restriction of h to its top S^{2p-1} either vanishes or has order p. A Serre spectral sequence computation (as done last year in an assignment) shows no element of order p exists in $\pi_{2p-1}(S^3)$, so the map is null-homotopic. Hence

$$C_h = C_g/S^2 \simeq Y := S^3 \vee S^{2p} \cup e^{2p+1}.$$

Let $f\colon S^{2p} \to S^3$ be the attaching map of this top $(2p+1)$-cell, its cone is $C_f = S^3 \cup_f e^{2p+1}$.

Then, there is a cellular map $C_g \to C_f$ inducing an isomorphism on cohomology in degrees 3 and $2p+1$, therefore

$$P^1 : H^3(C_f;\mathbb{Z}/p) \longrightarrow H^{2p+1}(C_f;\mathbb{Z}/p)$$

is still non-zero. Thus f has odd-primary Hopf invariant one, and in particular $f \neq 0$ in π^s_{2p-3}.

4. Constructing the operations

All of the previous sections relied on the Steenrod powers, but a priori, there is no reason to believe such cohomology operations exist. We give an outline of the construction given in [3, 4L] of the Steenrod squares and Steenrod powers in this section, while implicitly proving some of the properties (not all).

Fix a based CW–complex (X, x_0) and a prime p. Define

$$T\colon X^{\wedge p} \longrightarrow X^{\wedge p}, \quad T(x_1, \ldots, x_p) = (x_2, \ldots, x_p, x_1),$$

so T generates the usual $\mathbb{Z}/p$-action on the smash power. We can give $S^\infty = \bigcup_n S^{2n-1} \subset \mathbb{C}^\infty$ the free $\mathbb{Z}/p$-action obtained by rotating every complex coordinate through $2\pi/p$. The quotient is then the infinite lens space $L(\infty, p) := S^\infty/(\mathbb{Z}/p)$.
Let

$$\Gamma X = S^\infty \times_{\mathbb{Z}/p} X^{\wedge p}$$

be the orbit space for the diagonal action. Then, projection to S^∞ descends to a fibration

$$\tilde{\pi}\colon \Gamma X \longrightarrow L(\infty, p), \qquad \tilde{\pi}^{-1}(z) = X^{\wedge p}.$$

Since the cyclic action fixes the smash product basepoint x_0, the inclusion $S^\infty \times \{x_0\}$ induces a section $\tilde{i}\colon L(\infty, p) \hookrightarrow \Gamma X$. Collapsing this section gives

$$\Lambda X = \Gamma X / \tilde{i}\big(L(\infty, p)\big).$$

Every fiber $X^{\wedge p}$ meets the section in a single point, so $X^{\wedge p} \subset \Lambda X$ survives. Replacing S^∞ by S^1 in these constructions gives subspaces $\Gamma^1 X \subset \Gamma X$ and $\Lambda^1 X \subset \Lambda X$. All of these constructions are functorial in X since given some $f\colon X \to Y$, we have $Ff\colon FX \to FY$ for F any of these constructions.
Using the standard CW structures on $L(\infty, p)$ and S^∞, if the n-skeleton of X is S^n, then the pn-skeleton of ΛX is S^{pn}.
For $n > 0$ let $K_n = K(\mathbb{Z}/p, n)$ and write $\iota \in H^n(K_n)$ for the canonical generator. With $\mathbb{Z}/p$-coefficients the Künneth map gives $\widetilde{H}^*(X)^{\otimes p} \cong \widetilde{H}^*(X^{\wedge p})$, denote the image of $\alpha \in H^n(X)$ by $\alpha^{\otimes p}$.
Our aim now is to construct, for every α, a class $\lambda(\alpha) \in H^{pn}(\Lambda X)$ whose restriction to each fiber $X^{\wedge p}$ equals $\alpha^{\otimes p}$. By naturality, this reduces to producing $\lambda(\iota) \in H^{pn}(\Lambda K_n)$.

On $K_n^{\wedge p}$ the permutation T satisfies $T^*(\iota^{\otimes p}) = \iota^{\otimes p}$. Its restriction to the pn-skeleton

$$(S^n)^{\wedge p} = \big((S^1)^{\wedge n}\big)^{\wedge p} = S^{pn}$$

has degree $(-1)^{(p-1)n^2}$, hence is homotopic to the identity. Note that the case $p = 2$ uses $\pi_{2n}(K_{2n}) = \mathbb{Z}/2$, so is a little different from odd primes. Since $\pi_j(K_{pn}) = 0$ for $j > pn$, the homotopy extends over all cells and gives a map

$$\lambda\colon \Lambda K_n \longrightarrow K_{pn}$$

unique up to homotopy. Set $\lambda(\iota) := \lambda^*(\iota)$.

For a general $\alpha \in H^n(X)$ define

$$\lambda(\alpha) = (\Lambda\alpha)^*\,\lambda(\iota) \in H^{pn}(\Lambda X),$$

which restricts to $\alpha^{\otimes p}$ on every fiber.
We can then compose the diagonal map $L(\infty, p) \times X \hookrightarrow \Gamma X$ with the quotient $\Gamma X \to \Lambda X$ to get $\nabla\colon L(\infty, p) \times X \to \Lambda X$. Then, pulling back $\lambda(\alpha)$ gives

$$\nabla^*\big(\lambda(\alpha)\big) = \sum_j \omega_{(p-1)n-j} \otimes \theta_j(\alpha) \in H^{pn}(L(\infty, p)) \otimes H^*(X),$$

where ω_j denotes a fixed generator of $H^j\big(L(\infty, p)\big)$.
The maps $\theta_j\colon H^n(X) \to H^{n+j}(X)$ are natural cohomology operations and vanish for $j < 0$.

Then, for $p = 2$ we define $\mathrm{Sq}^j := \theta_j$. For odd p, things are bit trickier. One shows $\theta_j = 0$ unless $j = 2k(p-1)$ or $2k(p-1)+1$, then after applying normalising constant we define

$$P^k(\alpha) = (-1)^k a_n^{-1}\,\theta_{2k(p-1)}(\alpha), \quad a_n \neq 0,$$

so that $P^0 = \mathrm{id}$ and $P^k(\alpha) = \alpha^p$ whenever $|\alpha| = 2k$.
This completes the (sketch) construction of Steenrod squares and Steenrod powers from the fiberwise class $\lambda(\alpha)$.

References

[1] J. F. Adams. On the non-existence of elements of hopf invariant one. *Annals of Mathematics*, 72(1):20–104, 1960.

[2] J. F. Adams and M. F. Atiyah. k-theory and the hopf invariant. *Quarterly Journal of Mathematics*, 17(1):31–38, 1966.

[3] Allen Hatcher. *Algebraic Topology*. Cambridge University Press, 2002. Freely available online.