APPLICATIONS OF DIFFERENTIAL FORMS TO HOMOTOPY THEORY

TOBIN CARLTON VAN BUIZEN

Abstract. In this work we define many basics from homotopy theory, proving key results such as Hurewicz's theorem. We work towards more difficult cohomology and homotopy calculations. We finish with bringing differential forms in to study rational homotopy theory.

1. Introduction

This report delves into the essential concepts and tools of homotopy theory, beginning with the fundamental definitions of homotopy groups $\pi_k(X, x_0)$. A significant portion of the discussion centers around pivotal theorems such as Whitehead's Theorem - which gives a reason to care about homotopy theory. The Hurewicz Theorem is presented as a bridge between homotopy and homology, offering a method to compute homotopy groups using homological information under certain conditions. Through detailed examples and computations, including the use of spectral sequences, we illustrate the application of these theoretical tools to concrete problems.
Despite these powerful methods, the computation of torsion elements in homotopy groups remains notoriously challenging. The intricacies involved in determining torsion information highlight the limitations of traditional homotopy theoretic techniques when dealing with finite or complex topological spaces.

To address these challenges, we introduce rational homotopy theory, a field that focuses on the rationalisation of spaces to simplify the computation of homotopy groups. By considering homotopy groups tensored with the rational numbers, we can effectively isolate the free part of the homotopy groups, bypassing the complications introduced by torsion elements.
The culmination of this report is the exploration of Sullivan's minimal models, an algebraic framework that provides a systematic method for computing the rational homotopy groups of simply connected spaces. By constructing differential graded algebras that capture the essential homotopical information of a space, minimal models translate topological problems into algebraic ones. The big result demonstrated utilises differential forms (name of the course...) to say that for simply connected manifolds, the rational homotopy groups can be directly computed from the minimal model, with the dimension of $\pi_k(M) \otimes \mathbb{Q}$ corresponding to the number of generators in degree k of the minimal model.

2. Homotopy Theory Basics

The first point of business is to go through some basic definitions and theorems in homotopy theory. There are two main things discussed in this section are: first, Whitehead's theorem - which establishes conditions under which maps inducing isomorphisms on all homotopy groups lead to homotopy equivalences between spaces. Second, extending fiber bundles to fibrations which allows us to use the Leray-Serre spectral sequence for a wider class of applications.

Definition 2.1 (Homotopy group)**.** Let X be a topological space with a basepoint x_0. for $k \geq 1$, the kth homotopy group $\pi_k(X, x_0)$ of X based at x_0 is the homotopy classes of basepoint preserving maps from $S^k \to X$. We can also think of this as maps from $I^k \to X$ that map the faces to basepoint $\partial I^k \to x_0$

Remark 2.2 (Importance of basepoint)**.** Note, even if X is path connected, it can be ambiguous to just write $\pi_k(X)$.
Take X to be path connected and consider $\pi_k(X, x_0)$ and $\pi_k(X, y_0)$. Let $\overline{x}$ and $\overline{y}$ are the constant loops at x_0 and y_0 respectively. Notice that a path γ from x_0 to y_0 in X induces a map $F\colon \Omega_x X \to \Omega_y X$ given by $\psi \mapsto \gamma\psi\gamma^{-1}$ for any $\psi \in \Omega_x X$. This then induces a map of homotopy groups $\gamma_*\colon \pi_{k-1}(\Omega_x X, \overline{x}) \to \pi_{k-1}(\Omega_y X, \overline{y})$. Note that γ_* is an isomorphism with inverse $(\gamma^{-1})_*$. Let $[\alpha]$ be an element of $\pi_k(X, x_0)$, define a map F to be α on the bottom face of the cube I^{k+1} and γ on the vertical faces. We can extend the map F to the entire I^{k+1}. Its restriction to the top face represents $\gamma_*[\alpha]$. We can then see that γ_* depends only on the homotopy class of γ. When we take $x_0 = y_0$, the map $\gamma \mapsto \gamma_*$ can be thought of as a group action of $\pi_1(X, x_0)$ on $\pi_k(X, x_0)$. If this action is trivial, we can speak unambiguously about $\pi_k(X)$ (without reference to basepoint). When it is clear from context, we will write without reference to basepoint.

Corollary 2.3. *Let $[X, Y]$ denote the homotopy classes of continuous maps from X to Y (without reference to basepoint). We have a bijection $\pi_k(X, x_0)/\pi_1(X, x_0) \to [S^k, X]$.*

We know that if two spaces X, Y are homotopy equivalent, then they have isomorphic cohomology groups and rings. However, the converse implication does not hold. In order to have a sufficient condition for homotopy equivalences, we really need to extend to homotopy theory. This is shown through Whitehead's Theorem.

Theorem 2.4 (Whitehead)**.** *If $f\colon X \to Y$ is a continuous map between topological spaces that induces isomorphisms on all homotopy groups*

$$f_*\colon \pi_k(X) \xrightarrow{\cong} \pi_k(Y), \quad \text{for all } k \in \mathbb{N}$$

Then X and Y are homotopy equivalent.

Theorem 2.5. *A covering space $p\colon (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ induces isomorphisms*

$$p_*\colon \pi_n(\tilde{X}, \tilde{x}_0) \xrightarrow{\cong} \pi_n(X, x_0), \quad \text{for all } n \geq 2.$$

Proof idea. For surjectivity, apply the lifting criterion which implies every map $(S^n, s_0) \to (X, x_0)$ lifts to $(\tilde{X}, \tilde{x}_0)$ for simply connected S^n (so $n \geq 2$). For injectivity, use the homotopy lifting property (defined in Definition 2.7). ■

Remark 2.6 (Map in Whitehead's Theorem)**.** The isomorphism being induced by a map is important in Whitehead's theorem. We give a quick example of what can happen if the homotopy groups are isomorphic but not induced by a map.
Consider the following spaces: $X = S^2 \times \mathbb{R}P^3$, $Y = \mathbb{R}P^2 \times S^3$. Clearly $\pi_1(X) = \mathbb{Z}/2 = \pi_1(Y)$, and both X, and Y have the same universal cover $S^2 \times S^3$, so $\pi_k(X) \cong \pi_k(Y)$ for all $k \in \mathbb{N}$. But, we can show that X and Y are not homotopy equivalent. Recall that

$H_2(\mathbb{R}P^2) = 0$ while $H_2(\mathbb{R}P^3) = \mathbb{Z}/2$. Using this, by the Kunneth theorem for homology we compute $H_2(X)$:

$$\begin{aligned} H_2(X) &= H_2(S^2 \times \mathbb{R}P^3) \\ &\cong (H_1(S^2) \otimes H_1(\mathbb{R}P^3)) \oplus (H_0(S^2) \otimes H_2(\mathbb{R}P^3)) \oplus (H_2(S^2) \otimes H_0(\mathbb{R}P^3)) \\ &\cong (0 \otimes \mathbb{Z}/2) \oplus (\mathbb{Z} \otimes 0) \oplus (\mathbb{Z} \otimes \mathbb{Z}) \\ &\cong \mathbb{Z} \end{aligned}$$

So $H_2(X) \cong \mathbb{Z}$. On the other hand, we compute $H_2(Y)$:

$$\begin{aligned} H_2(Y) &= H_2(\mathbb{R}P^2 \times S^3) \\ &\cong (H_1(\mathbb{R}P^2) \otimes H_1(S^3)) \oplus (H_0(\mathbb{R}P^2) \otimes H_2(S^3)) \oplus (H_2(\mathbb{R}P^2) \otimes H_0(S^3)) \\ &\cong (\mathbb{Z}/2 \otimes 0) \oplus (\mathbb{Z} \otimes 0) \oplus (0 \otimes \mathbb{Z}) \\ &\cong 0 \end{aligned}$$

Since $H_2(X) \neq H_2(Y)$, X and Y are not homotopy equivalent. This shows that we did need a map to induce the isomorphisms on homotopy groups.

In order to do some of the computations that we want to do, we need to extend the idea of fiber bundles a little bit.

Definition 2.7 (Fibration)**.** A map $\pi\colon E \to X$ is called a fibration if it satisfies the homotopy lifting property. The homotopy lifting property: given a map $f\colon Y \to E$ and a homotopy F_t of $\overline{f} = \pi \circ f$ in X, there is a homotopy $\tilde{F}_t$ of f in E which covers F_t. This is to say that $\pi \circ \tilde{F}_t = F_t$. This property may be expressed in terms of the diagram below.

$$\begin{array}{ccc} Y & \xrightarrow{\;f\;} & E \\ {\scriptstyle (id,0)}\big\downarrow & \overset{\tilde{F}_t}{\nearrow} & \big\downarrow{\scriptstyle \pi} \\ Y \times I & \xrightarrow[F_t]{} & X \end{array}$$

Note, as long as the map $\pi\colon E \to X$ has the property that $H^k(\pi^{-1}U) \simeq H^k(F)$ for some fixed space F and any contractible open U, then $E_2 = H_\delta H_d(K)$ will be the same as for the fiber bundle. In particular, if $E \to X$ is a fibration, this holds. This means that we can use the Leray-Serre spectral sequence for any fibration.

Theorem 2.8. *Let $p : E \to B$ be a basepoint preserving fibration with fiber F. This means that we have the map $i\colon F \to E$. Then there is a long exact sequence of homotopy groups*

$$\cdots \longrightarrow \pi_k(F) \xrightarrow{\;i_*\;} \pi_k(E) \xrightarrow{\;p_*\;} \pi_k(B) \xrightarrow{\;\partial\;} \pi_{k-1}(F) \longrightarrow \cdots \longrightarrow \pi_0(B) \longrightarrow 0$$

Proof sketch. Let $I^{k-1} \subset I^k$ be the inclusion. A map $\alpha\colon I^k \to B$ representing an element of $\pi_k(B)$ may be regarded a a homotopy of $\alpha|_{I^{k-1}}$ in B. Let the constant map $c\colon I^{k-1} \to E$ from I^{k-1} to the basepoint of F be the map that covers $\alpha|_{I^{k-1}}\colon (x_1, x_2, \dots x_{k-1}, 0) \to B$. By the homotopy lifting property, there is a homotopy $\tilde{\alpha}\colon I^k \to E$ which covers α and such that $\tilde{\alpha}|_{I^{k-1}} = c$. Then, $\partial[\alpha]$ is the class of the map $\tilde{\alpha}\colon (x_1, \dots x_{k-1}, 1) \to F$. Thus $\partial[\alpha]$ is well defined. ∎

Definition 2.9. Let A be an abelian group. We say that X is an Eilenberg-Maclane space, written $K(A,n)$ if

$$\pi_i(X) = \begin{cases} A & i = n \\ 0 & \text{else} \end{cases}$$

Note, A need not be abelian for $n = 1$. Also note that if we consider only CW complexes, $K(A,n)$ is unique up to homotopy equivalence.

3. Computing Homotopy Information

With the very basic theory covered, we move into more computational tools. We state and prove Hurewicz's theorem which connects homotopy and homology information. We also do a few examples (exercises from the book) to illustrate the difficulty of some of the computations in homotopy theory.
Before any of that, we give an important trick for many homotopy calculations. We know that the Serre spectral sequence requires a fibration, but just how restrictive is this? We will show that this is not particularly restrictive at all.
To start, let X be some space and Y some path-connected space. Then we will show that we can treat a continuous map $f\colon X \to Y$ as inclusion, and then as a fibration.
We construct the mapping cylinder of f:

$$M_f = (X \times I) \cup Y/((x,1) \sim f(x))$$

Clearly M_f has the same homotopy type of Y, but also we see that $X \subset M_f$. So for the purposes of homotopy theory, we can treat f as an inclusion of X into $M_f \simeq Y$.
Next, with this trick, assume that X is a subspace of Y. Then define E to be the space of all paths in Y with starting point in X. Clearly $E \simeq X$ by shrinking every path to its starting point. We could do the opposite and project every path to its end point, this gives a fibration

$$\begin{array}{ccccc} \Omega_*^X & \longrightarrow & E & \overset{\simeq}{\text{——}} & X \\ & & \downarrow & & \\ & & Y & & \end{array}$$

Thus (up to homotopy equivalence), $f\colon X \to Y$ is a fibration.

The notation Ω has been used with no mention. We now define path and loop spaces.

Definition 3.1. Path space of X is the space PX consisting of all the paths in X with initial point $*$:

$$PX = \{\text{maps } \mu\colon I \to X \mid \mu(0) = *\}$$

Loop space of X is the space ΩX consisting of all the loops in X with initial point $*$:

$$\Omega X = \{\mu\colon I \to X \mid \mu(0) = \mu(1) = *\}$$

With these definitions, there is a natural projection $p\colon PX \to X$ given by the endpoint of a path. From this, we get the path fibration: $\Omega X \to PX \to X$.

Proof. Let U be a contractible open containing x_0. There is the inclusion $i\colon p^{-1}(x_0) \to p^{-1}(U)$. Since we can contract U to p, we get a map the other way $r\colon p^{-1}(U) \to p^{-1}(x_0)$. It is easy to see that r and i are homotopy inverses. Also, if x_1 and x_2 are two points in the same path component of X, then a path from x_1 to x_2 induces a homotopy equivalence $p^{-1}(x_1) \simeq p^{-1}(x_2)$. Thus, all fibers have the homotopy type of $p^{-1}(x_0)$, which is exactly ΩX. ■

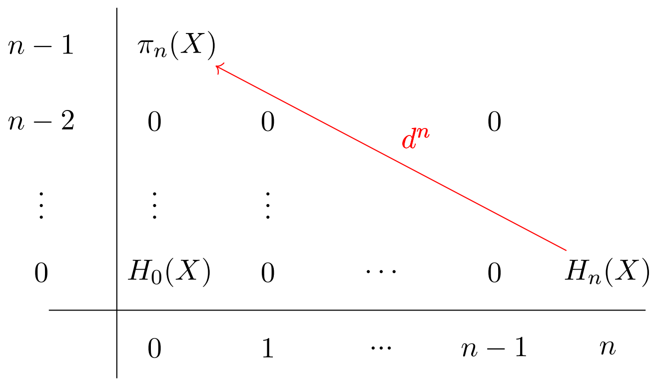
Note that $PX \simeq *$, since we can retract every path back to $*$.

Proposition 3.2. *$\pi_{k-1}(\Omega X) = \pi_k(X)$ for $k \geq 2$.*

Proof idea. View a map from I^k to X as a map from I^{k-1} to ΩX. ■

Theorem 3.3 (Hurewicz)**.** *Let X be a simply connected path-connected CW complex. Then the first nontrivial homotopy and homology occur in the same dimension and are equal. That is, given $n \in \mathbb{N}$ with $n \geq 1$, and $\pi_k(X) = 0 = H_k(X)$ for $2 \leq k < n$, then $\pi_n(X) = H_n(X)$. .*

Proof. We prove by induction. Consider the base case with $n = 1$. In the case that X is a path-connected CW complex, $\pi_1(X)_{\text{ab}} = H_1(X)$. It is easy to show that $\pi_k(X)$ is abelian for $k > 1$, so this serves as our base case. Now, assume the Hurewicz theorem for $n - 1$, and we show it is true for n. Since X is $(n-1)$-connected, ΩX is $(n-2)$-connected. Then, applying the hypothesis to ΩX, we get $\pi_{n-1}(\Omega X) \cong H_{n-1}(\Omega X)$. Thus, $\pi_n(X) \cong H_{n-1}(\Omega X)$. Now, by the induction hypothesis again, we get the following page of the spectral sequence.



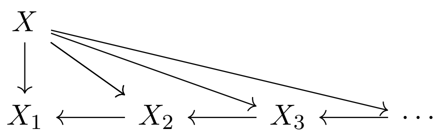
Since $PX \simeq *$, everything (outside of $H_0(X)$) has to be killed, and the only differential that could kill $H_{n-1}(\Omega X) \cong \pi_n(X)$ is the d^n drawn (since the other columns are filled with zeros by the induction hypothesis). Thus, $d^n : H_n(X) \to H_{n-1}(\Omega X) \cong \pi_n(X)$ must be an isomorphism. Hence, $H_n(X) \cong \pi_n(X)$. This proves the Hurewicz theorem. ■

For the final computation of this section, we need a slightly more general version of the Hurewicz theorem, a couple more definitions and some small theorems.

Definition 3.4. A Serre class $\mathscr{C}$ is a class of abelian groups closed under taking subgroups, quotients and extensions [3, p. 532].

Theorem 3.5. *Let X be a path-connected space X with abelian fundamental group (this includes 0). If $\pi_k(X) \in \mathscr{C}$ for $k < n$ then $\pi_n(X) \cong H_n(X)$ mod $\mathscr{C}$ [3, p. 533].*

Definition 3.6. Let X be a path-connected space. A Postnikov tower for X is a commutative diagram as below such that:



- The map $X \to X_n$ induces an isomorphism on π_k for $k \leq n$,
- $\pi_k(X_n) = 0$ for $k > n$,
- The map $X_n \to X_{n-1}$ is a fibration with fiber a $K(\pi_n(X), n)$.

Remark 3.7. This third part of the definition is actually a consequence of the first two, but this is unimportant.

Lemma 3.8. *For X a connected CW complex, X has a Postnikov tower that is unique up to homotopy equivalence [2, p. 410].*

Theorem 3.9. *Given a fibration $F \to E \to B$, we can produce a Puppe sequence*

$$\cdots \to \Omega^2 B \to \Omega F \to \Omega E \to \Omega B \to F \to E \to B$$

Any two consecutive maps in this sequence form a fibration (up to homotopy equivalence) and all maps to the left of ΩB are obtained by applying Ω to the later maps.

Using all this new theory, we start with a little warm-up computation before the big one.

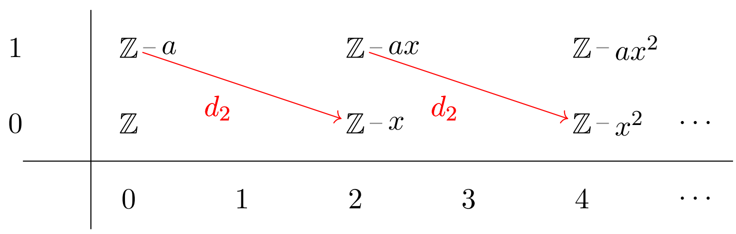
Example: Exercise 18.8. Let $X = K(\mathbb{Z}/q, 1)$. We want to compute $H^*(X;\mathbb{Z})$ and $H^*(X;\mathbb{Z}/p)$ for p prime. First, we know by the definition of the infinite lens space, $L(\infty, q) \cong K(\mathbb{Z}/q, 1)$. With this, since S^∞ is the universal cover of X, we then have the fibration

$$\mathbb{Z}/q \to S^\infty \to K(\mathbb{Z}/q, 1)$$

However, we cannot use this in the Serre spectral sequence because X is (by definition) not simply connected. To fix this, we mod out the spaces by $\mathbb{Z}/q$, and the action $S^1/(\mathbb{Z}/q) \cong S^1$ on X. We know that $S^\infty/S^1 \cong \mathbb{C}P^\infty$, so we get the fibration

$$S^1 \to K(\mathbb{Z}/q, 1) \to \mathbb{C}P^\infty$$

Now we can use the Serre spectral sequence to compute cohomology.



The only possible nontrivial differentials are the d_2 drawn (repeating to the right). We want to determine what these maps are. To do this, first we know that $\pi_1(L(\infty, q)) = \mathbb{Z}/q$. Since this is abelian, by Hurewicz, $H_1(L(\infty, q)) = \mathbb{Z}/q$. Then by the universal coefficient theorem for cohomology, $H^2(L) = \mathbb{Z}/q \oplus (\text{something free})$, so $d_2(a) = qx$.

Thus $H^2(L) = \mathbb{Z}/q$. Then by the multiplicative structure (given by cup product), we get

$$H^k(K(\mathbb{Z}/q,1);\mathbb{Z}) = \begin{cases} \mathbb{Z} & k=0 \\ \mathbb{Z}/q & k>0 \text{ odd} \\ 0 & \text{else.} \end{cases}$$

Now to get with $\mathbb{Z}/p$ coefficients, we again use the universal coefficient theorem.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ext}(H_1(L(\infty,q)),\mathbb{Z}/p) & \longrightarrow & H^2(L(\infty,q)) & \longrightarrow & \mathrm{Hom}(H_2(L(\infty,q)),\mathbb{Z}/p) & \longrightarrow & 0 \\ 0 & \longrightarrow & \mathrm{Ext}(\mathbb{Z}/q,\mathbb{Z}/p) & \longrightarrow & H^2(L(\infty,q)) & \longrightarrow & \mathrm{Hom}(0,\mathbb{Z}/p) & \longrightarrow & 0 \\ 0 & \longrightarrow & \mathrm{Ext}(\mathbb{Z}/q,\mathbb{Z}/p) & \longrightarrow & H^2(L(\infty,q)) & \longrightarrow & 0 & & \end{array}$$

Then $\mathrm{Ext}(\mathbb{Z}/q,\mathbb{Z}/p) = \mathbb{Z}/\gcd(p,q)$, thus $H^2(L;\mathbb{Z}/p) = \mathbb{Z}/\gcd(p,q)$. Then by the multiplicative structure, we get

$$H^k(K(\mathbb{Z}/q,1);\mathbb{Z}/p) = \begin{cases} \mathbb{Z}/p & k=0 \\ \mathbb{Z}/\gcd(p,q) & k>0 \text{ even} \\ 0 & \text{else.} \end{cases}$$

■

That concludes our warmup exercise. Now we give an example that shows the difficulty of computing torsion information in homotopy groups.

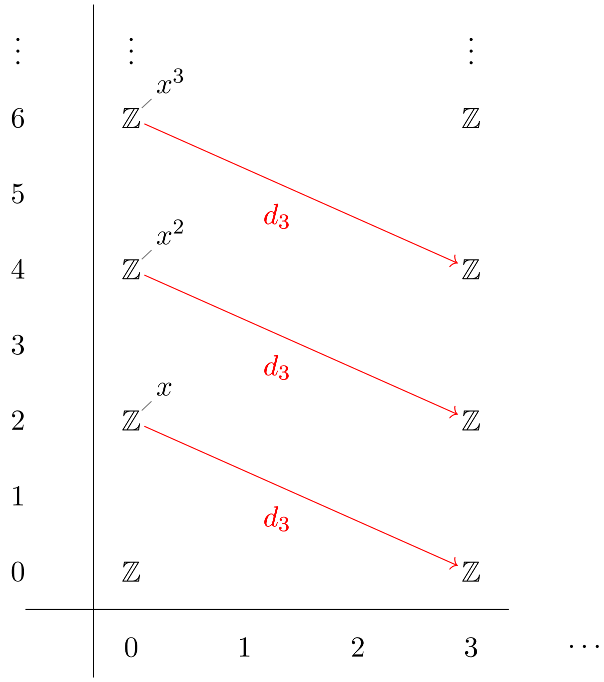
Example: Exercise 18.24. We want to find the least k such that $\pi_k(S^3)$ has p torsion. Starting with the obvious map $S^3 \to K(\mathbb{Z},3)$ (map basepoint to basepoint, then extend so that S^3 wraps around the $K(\mathbb{Z},3)$) inducing an isomorphism on π_3. We can turn this map into a fibration with fiber F:

$$F \longrightarrow S^3 \longrightarrow K(\mathbb{Z},3)$$

This means that F is 3-connected and $\pi_i(F) \cong \pi_i(S^3)$ for $i \geq 3$. Then we use the Puppe sequence to turn the map $F \to S^3$ (or at least something homotopic to F) into a fibration:

$$\Omega K(\mathbb{Z},3) = K(\mathbb{Z},2) \longrightarrow F \longrightarrow S^3$$

We know that $\mathbb{C}P^\infty$ is an example of $K(\mathbb{Z},2)$ and $H^*(\mathbb{C}P^\infty;\mathbb{Z}) \cong \mathbb{Z}[x]$, $|x|=2$. Combining this with the cohomology of S^3, we use the Serre spectral sequence applied to this fibration.



Let $\iota \in H^3(S^3)$. We need F to be $3-$connected, so by Hurewicz, we need $d_3(x)$ to be an isomorphism and so, $d_3(x) = \iota$. Then, recall that d satisfies a Leibniz rule, so $d_3(x^n) = n\iota x^{n-1}$. This means that the image of each differential from x^n is $\mathbb{Z}/n$. From this, we get that the cohomology of F is entirely given by the 3 column. Thus,

$$H^i(F) = \begin{cases} \mathbb{Z}/n & i = 2n+1, n > 0 \\ 0 & \text{else} \end{cases}$$

Then the universal coefficient theorem shifts torsion down a degree, hence,

$$H_i(F) = \begin{cases} \mathbb{Z}/n & i = 2n, n > 0 \\ 0 & \text{else} \end{cases}$$

To finish off, we use the Serre class Hurewicz theorem, where the Serre class is $\mathscr{C}$: finite abelian groups with torsion prime to fixed prime p. Since p is prime, all of the $H_i(F)$ will have no p torsion until $i = 2p$. So, then we get that $H_{2p}(F) \cong \pi_{2p}(F) \cong \pi_{2p}(S^3)$. Hence, the least k is $2p$. ■

It is important to note that the reasoning here worked only for S^3. To extend to S^n for higher n, we need much more advanced techniques. This shows just how tricky working out torsion of homotopy groups can be.

4. Rationalise and Differential Forms

At the end of the previous section we saw how difficult it can be to determine basic torsion information for homotopy groups. We want a way to isolate the free part of homotopy groups, because perhaps this can give us a richer understanding more easily. This leads us to rational homotopy groups, and ways to compute them.

In rational homotopy theory, Sullivan minimal models provide an algebraic framework to study the rational homotopy type of topological spaces, especially simply connected spaces of finite type. They translate topological problems into algebraic ones by associating a differential graded algebra to a space, capturing its rational homotopy information.

Definition 4.1. A differential graded algebra (DGA) is a graded algebra $A = \bigoplus_{i\geq 0} A^i$ equipped with a differential $d : A^i \to A^{i+1}$ such that $d^2 = 0$ and the (graded) Leibniz rule holds: $d(ab) = d(a)b + (-1)^{|a|}ad(b)$ for homogeneous elements $a \in A^i$.

Such an algebra is free if it satisfies no relations other than those of associativity and graded commutativity. $\Lambda(x_1, \dots, x_k)$ is the free algebra on $x_1, \dots, x_k$. This is the tensor product of the polynomial algebra on its even dim generators and exterior algebra on odd dim generators.

An element in $a \in A$ is decomposable if it is a sum of products of positive elements in A, that is: $a \in A^+ \cdot A^+$, $A^+ = \bigoplus_{i>0} A^i$.

Definition 4.2. A DGA $\mathscr{M}$ is called a minimal model for A if

- $\mathscr{M}$ is free,
- there is a chain map $f : \mathscr{M} \to A$ which induces an isomorphism on cohomology,
- the differential of a generator is either zero or decomposable.

Theorem 4.3. *Let M be a simply connected manifold and $\mathscr{M}$ its minimal model. Then the dimension of the vector space $\pi_k(M) \otimes \mathbb{Q}$ is the number of generators of the minimal model in dimension k.*

The construction of a minimal model $(\mathscr{M}, d)$ for a simply connected X proceeds inductively by adding generators to kill cohomology classes the prevent $\mathscr{M}$ from being quasi-isomorphic to A. This gives an easy way of computing the rational homotopy groups of manifolds. Note that this computes the dimension of the free part of the homotopy of manifolds (not the torsion).

Definition 4.4. A space X is said to be formal if its minimal model $(\mathscr{M}, d)$ is quasi-isomorphic to its cohomology ring $H^*(X; \mathbb{Q})$.

Lemma 4.5. *The rational homotopy type of a formal space X is determined solely by its cohomology ring $H^*(X; \mathbb{Q})$.*

Example 4.6. $\mathbb{C}P^n$ is formal for $n \in \mathbb{N}$. The minimal model is generated by a single generator in degree 2, with trivial differential. This is the same as $H^*(\mathbb{C}P^n; \mathbb{Q})$.

It can be difficult to determine whether a space X is formal or not. One method is to look at higher order cohomology operations, such as the Massey product. However, it is important to note that these operations can detect if a space is not formal, but not definitively if it is formal. This is similar to how cohomology can detect if spaces are not homotopy equivalent, but not definitely if they are.
We do an example to show how powerful and easy to work with minimal models are (and to show that differential forms came into this report).

Example. We look at computing the first few rational homotopy groups of $X = S^2 \vee S^2$. The first observation to make here is that X has the same homotopy type as $Y = \mathbb{R}^3 \setminus \{p_1, p_2\}$. This means that we can just construct a minimal model $\mathscr{M}$ for Y.
Now, we want to construct two closed 2-forms that generate the cohomology $H^2_{DR}(Y)$.

To do this, take small spheres S_1 and S_2 around p_1 and p_2 respectively. Let ω_1 be a bump form that integrates to 1 with support concentrated near the north pole of S_1. Let ω_2 be a bump form that integrates to 1 with support concentrated near the south pole of S_2. There are then natural projection maps $\pi_{p_1}: Y \to S_1$ and $\pi_{p_2}: Y \to S_2$. We then define $\overline{x}_1 = \pi_{p_1}^*\omega_1$ and $\overline{x}_2 = \pi_{p_2}^*\omega_2$. These $\overline{x}_1$ and $\overline{x}_2$ generate $H^2_{DR}(Y)$ and satisfy $\overline{x}_1^2 = \overline{x}_1\overline{x}_2 = \overline{x}_2^2 = 0$.
That was the extent of the geometry. We now construct the minimal model using this information. $\mathscr{M}$ must have two generators x_1, x_2 in dimension 2 mapping to $\overline{x}_1$ and $\overline{x}_2$. We know that the cohomology of Y should be trivial above dimension 2, and only two generators in dimension 2. This means, we need to kill off the relation terms living in dimension 4 that equaled 0: x_1^2, x_2^2, x_1x_2. These are closed forms but should not represent a cohomology, so they must die. They are not multiples of each other, so we need three generators y_1, y_2, y_3 in dimension 3 to kill them off with:

$$dy_1 = x_1^2, \quad dy_2 = x_1x_2, \quad dy_3 = x_2^2$$

So far, this means that our chain map $f: \mathscr{M} \to Y$ is given by:

$$x_1 \mapsto \overline{x}_1, \quad x_2 \mapsto \overline{x}_2, y_1, y_2, y_3 \mapsto 0$$

We then look at linear combinations of our generators and apply differentials to them to see what has been added and what needs to be killed off. We go through the full details of the differentials of the elements in dimension are 5:

$$\begin{aligned}
&[x_1 \text{ closed}] && d(y_1x_1) = d(y_1)x_1 - y_1d(x_1) = x_1^3 - 0 = x_1^3\\
&[x_2 \text{ is closed}] && d(y_1x_2) = d(y_1)x_2 - y_1d(x_2) = x_1^2x_2 - 0 = x_1^2x_2\\
&&& d(y_2x_1) = d(y_2)x_1 - y_1d(x_1) = x_1^2x_2 - 0 = x_1^2x_2\\
&&& d(y_2x_2) = d(y_2)x_2 - y_1d(x_2) = x_1x_2^2 - 0 = x_1x_2^2\\
&&& d(y_3x_1) = d(y_3)x_1 - y_1d(x_1) = x_1x_2^2 - 0 = x_1x_2^2\\
&&& d(y_3x_2) = d(y_3)x_2 - y_1d(x_2) = x_2^3 - 0 = x_2^3.
\end{aligned}$$

We note that the second and third lines are equal and the fourth and fifth lines are equal. This means that $d(y_1x_2) - d(y_2x_1) = d(y_1x_2 - y_2x_1) = 0$ and $d(y_2x_2 - y_3x_1) = 0$ and we have found two closed forms in dimension 5. These must be killed, so we need two elements z_1, z_2 in dimension 4 to kill them:

$$dz_1 = y_1x_2 - y_2x_1, \quad dz_2 = y_2x_2 - y_3x_1$$

We continue this process up to dimension 6 in order to verify the table given in Bott & Tu [1, p. 265].
Now we look at the differential of the elements in dimension 6 (in less detail):

$$\begin{aligned}
d(z_1x_1) &= y_1x_1x_2 - y_2x_1^2, & d(z_1x_2) &= y_1x_2^2 - y_2x_1x_2,\\
d(z_2x_1) &= y_2x_1x_2 - y_3x_1^2, & d(z_2x_2) &= y_2x_2^2 - y_3x_1x_2,\\
d(y_1y_2) &= y_2x_1^2 - y_1x_1x_2, & d(y_2y_3) &= y_3x_1x_2 - y_2x_2^2,\\
d(y_1y_3) &= y_3x_1^2 - y_1x_2^2.
\end{aligned}$$

From here, we can deduce that $z_1x_1 + y_1y_2$, $z_2x_2 + y_2y_3$ and $z_1x_2 + z_2x_1 + y_1y_3$ are all closed. These must be killed, but none of the elements we already have in dimension 5 map to these, so we need three new elements a_1, a_2, a_3 in dimension 5 to kill them:

$$da_1 = z_1x_1 + y_1y_2, \quad da_2 = z_2x_2 + y_2y_3, \quad da_3 = z_1x_2 + z_2x_1 + y_1y_3.$$

Now we go beyond the book to verify the table given - we look at the differential of the elements in dimension 7 (orders presented in suspicious ways on purpose):

$$\begin{aligned}
d(y_1x_1^2) &= x_1^4, & d(a_1x_1) &= z_1x_1^2 + y_1y_2x_1, & d(a_1x_2) &= z_1x_1x_2 + y_1y_2x_2,\\
d(y_1x_1x_2) &= x_1^3x_2, & d(a_2x_1) &= z_2x_1x_2 + y_2y_3x_1, & d(a_2x_2) &= z_2x_2^2 + y_2y_3x_2,\\
d(y_1x_2^2) &= x_1^2x_2^2, & d(a_3x_1) &= z_1x_1x_2 + z_2x_1^2 + y_1y_3x_1, & d(a_3x_2) &= z_1x_2^2 + z_2x_1x_2 + y_1y_3x_2,\\
d(y_2x_1^2) &= x_1^3x_2, & d(z_1y_1) &= y_1^2x_2 - y_1y_2x_1 + z_1x^2, & d(z_1y_2) &= y_1y_2x_2 - y_2^2x_1 + z_1x_1x_2,\\
d(y_2x_1x_2) &= x_1^2x_2^2, & d(z_1y_3) &= y_1y_3x_2 - y_2y_3x_1 + z_1x_2^2, & d(y_1z_2) &= z_2x_1^2 - y_1y_2x_2 + y_1y_3x_1,\\
d(y_2x_2^2) &= x_1x_2^3, & d(z_2y_2) &= y_2^2x_2 - y_2y_3x_1 + z_2x_1x_2, & d(z_2y_3) &= y_2y_3x_2 - y_3^2x_1 + z_2x_2^2,\\
d(y_3x_1^2) &= x_1^2x_2^2, & d(y_2x_1x_2) &= x_1x_2^3, & d(y_3x_2^2) &= x_2^4.
\end{aligned}$$

From here, we can deduce that: $y_1x_2^2 - y_2x_1x_2$, $y_3x_1^2 - y_2x_1x_2$, $y_1x_2^2 - y_3x_1^2$, $y_2x_2^2 - y_3x_1x_2$, $a_3x_2 - a_2x_1 - z_1y_3$ and $a_3x_1 - y_1z_2 - a_1x_2$ are all closed. These must be killed. One can check that no other linear combination gives 0 (looking at $y_j^2x_i$ only occurring once, so cannot die). This means that we need six new elements $b_1, b_2, b_3, b_4, b_5, b_6$ to kill them:

$$\begin{aligned}
d(b_1) &= y_1x_2^2 - y_2x_1x_2, & d(b_2) &= y_3x_1^2 - y_2x_1x_2, & d(b_3) &= y_1x_2^2 - y_3x_1^2,\\
d(b_4) &= y_2x_2^2 - y_3x_1x_2, & d(b_5) &= a_3x_2 - a_2x_1 - z_1y_3, & d(b_6) &= a_3x_1 - y_1z_2 - a_1x_2.
\end{aligned}$$

This matches with the result that Bott & Tu claim in their book [1, p. 265]. This process could continue. ■

This example exhibits the (relative) ease of these computations to construct a minimal model. Albeit, there is some tedious bookkeeping required, but there is no difficult mathematics involved.

Obtaining this rational homotopy theory information from purely homotopy theoretic techniques is quite challenging. This last section shows a strong application of differential forms and the theory learned in our class to homotopy theory. The use of differential forms makes determining this information much more routine.

References

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[2] Allen Hatcher. *Algebraic Topology*. Cambridge University Press, Cambridge, 2002.

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